MAT354 — Midterm handout: Chapters I–II (course notes 1–11, figures omitted)

October 13, 2026

Note 1 — Introduction: Coming Attractions

1. Fundamental Theorem of Algebra

On the invention of Complex numbers:

Consider a cubic equation of the form $x^3 + px + q = 0$. Using the famous Cardano's formula (from 1545):

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Is there only one root? No, there are always three complex cubic roots!

Example: Take the equation $x^3 - 15x = 4$, which means $p = -15$ and $q = -4$. We know a real root is $x = 4$. However, applying Cardano's formula gives:

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

What is wrong here? We need to take all possible values of the cubic roots to find the real solution.

The Fundamental Theorem of Algebra

Every (non-constant) polynomial with complex coefficients has a root. In fact, if it has degree n, it has at least n roots, counting multiplicity.

We will have at least three proofs of this Theorem!

2. Differentiation: Real vs. Complex

There are profound differences between differentiability in the real numbers and the complex numbers.

The Real Case

- **Definition:**
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
- **Limited Differentiability:** A function can be differentiable only once. Example: $f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$
- **Taylor Series Disconnect:** Even infinitely differentiable functions can have nothing to do with their Taylor series. Example: $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Here $f^{(n)}(0) = 0\ \forall n$.
- **Analyticity:** Real analytic functions equal their Taylor series: $f(x) = \sum a_n (x - x_0)^n$.

Hierarchy: Differentiable $\not\Rightarrow$ Infinitely Differentiable $\not\Rightarrow$ Analytic

The Complex Case

- **Definition:**
$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$
Functions satisfying this are **analytic**.
- **Infinite Differentiability:** Any function differentiable at a neighborhood of a point is automatically *infinitely differentiable*.
- **Taylor Series Equivalence:** Differentiable functions are automatically equal to the sum of their Taylor series:
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!}$$

Hierarchy: Differentiable $\Longrightarrow$ Infinitely Differentiable $\Longrightarrow$ Analytic

3. Series and Singularities

The behavior of power series highlights the necessity of the complex plane to understand singularities.

- **Real Singularity:**
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$
This function has an obvious singularity at $x = 1$.
- **Complex Singularity:**
$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x| < 1$$
While there are *no real singularities* for this function, the series still has a restricted radius of convergence because of **complex singularities** at $z = \pm i$.

4. Antiderivatives: Real vs. Complex

The Real Case

Any continuous function has an antiderivative: $F'(x) = f(x)$. By the Fundamental Theorem of Calculus (FTC):

$$F(x) = \int_a^x f(t)dt$$

The Complex Case

Only differentiable functions have antiderivatives.

$$F(z) = \int_{z_0}^{z} f(w)dw$$

This is a line integral. Over which path? For a valid antiderivative, it must be *path-independent:* over a closed loop γ: $\oint_\gamma f(z)dz = 0$.

Real Differentiability in $\mathbb{R}^2$ vs. Complex Differentiability

Real differentiability of f at z (viewing $f : \mathbb{R}^2 \to \mathbb{R}^2$) means

$$\big|f(z+h) - f(z) - T(h)\big| \longrightarrow 0 \quad \text{(relative to } |h|\text{), as } h \to 0,$$

for *some* $\mathbb{R}$-linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ – i.e. the Jacobian of f, with no further constraint on its four real entries. Writing $f = u + iv$, $z = x + iy$, this Jacobian is the 2×2 real matrix

$$T = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix},$$

four independent real numbers – generically, T need not have any special structure.

Complex differentiability asks for the *same* linear approximation, but with T forced to be **multiplication by a single complex number** λ:

$$\frac{\big|f(z+h) - f(z) - \lambda h\big|}{|h|} \longrightarrow 0, \qquad \lambda = f'(z).$$

Maps $h \mapsto \lambda h$ (rotation-scalings) are a very rigid 2-parameter family sitting inside all 4-parameter real-linear maps $\mathbb{R}^2 \to \mathbb{R}^2$. Demanding the derivative take this restricted form *is* the Cauchy–Riemann equations – and this extra rigidity is exactly what forces analytic functions to be infinitely differentiable.

5. Conformal Maps and Riemann Theorem

Conformal Maps: A special and highly important class consists of injective analytic functions. A map $\varphi : \Omega_1 \to \Omega_2$ is conformal if it is an angle- and orientation-preserving map.

Riemann Mapping Theorem

If Ω is a simply-connected domain (a domain without holes), and $\Omega \neq \mathbb{C}$, then there exists a conformal bijection between $\mathbb{D}$ and Ω.

Note 2a — Complex Plane: Vectors, Conjugates, and the Field Axioms

1. Complex Numbers as Vectors

A complex number $z = a + ib$ can be mapped to a vector $\binom{a}{b}$ in the Cartesian plane $\mathbb{R}^2$.

Standard notation. The real part is $a = \mathrm{Re}(z)$ and the imaginary part $b = \mathrm{Im}(z)$. The complex plane $\mathbb{C}$ is $\mathbb{R}^2$ equipped with specific rules for addition and multiplication: the real line $\mathbb{R}$ is identified with vectors $\binom{a}{0}$ on the x-axis, while purely imaginary numbers lie along the y-axis.

Addition behaves exactly like standard vector addition: $(a+ib)+(c+id) = (a+c)+i(b+d)$, i.e. $\binom{a}{b} + \binom{c}{d} = \binom{a+c}{b+d}$.

Multiplication is defined algebraically as $(a+ib)(c+id) = (ac-bd) + i(ad+bc)$, i.e.

$$\binom{a}{b} \cdot \binom{c}{d} = \binom{ac-bd}{ad+bc}, \qquad \text{so in particular } \binom{0}{1} \cdot \binom{0}{1} = \binom{-1}{0}.$$

2. Absolute Value and Conjugate

The Complex Conjugate

For $z = a + ib$, the **complex conjugate** is $\bar{z} := a - ib$, the reflection with respect to the *real line*.

Component formulas

$$\mathrm{Re}(z) = \frac{z+\bar{z}}{2}, \qquad \mathrm{Im}(z) = \frac{z-\bar{z}}{2i} = -\frac{i}{2}(z-\bar{z}).$$

Example (line equation). A line $ax + by = c$ (real notation, $a^2 + b^2 \neq 0$) becomes, substituting $x = \dfrac{z+\bar{z}}{2}$ and $y = \dfrac{z-\bar{z}}{2i}$,

$$a \cdot \frac{z+\bar{z}}{2} + b \cdot \frac{z-\bar{z}}{2i} = c, \qquad \text{i.e.} \qquad \frac{a-ib}{2}\, z + \frac{a+ib}{2}\, \bar{z} = c.$$

Writing $\mu := \dfrac{a-ib}{2}$ (so $\mu \neq 0$ and $\bar{\mu} = \dfrac{a+ib}{2}$), every line in $\mathbb{C}$ takes the compact form

$$\mu z + \bar{\mu}\bar{z} = c, \qquad \mu \neq 0,\ c \in \mathbb{R}.$$

Basic properties. $\overline{\bar{z}} = z$, $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\bar{w}$, and (for $w \neq 0$) $\overline{z/w} = \bar{z}/\bar{w}$.

Absolute Value (Modulus)

$|z|^2 = x^2 + y^2 = (x+iy)(x-iy) = z\bar{z} = |\bar{z}|^2$. $|z|$ is the distance from z to 0.

Properties

Multiplicative: $|zw| = |z||w|$. *Proof:* $|zw|^2 = zw\overline{zw} = zw\bar{z}\bar{w} = z\bar{z}\,w\bar{w} = |z|^2|w|^2$.
Triangle inequality: $|z+w| \le |z| + |w|$. *Proof:* $|z+w|^2 = (z+w)\overline{(z+w)} = |z|^2 + |w|^2 + z\bar{w} + \bar{z}w = |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\bar{w})$. Since $\mathrm{Re}(z\bar{w}) \le |z\bar{w}| = |z||w|$, this is $\le |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2$.

Scalar product

Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$. Then we have the following complex expression for the *scalar product*

$$\left\langle \binom{a_1}{b_1}, \binom{a_2}{b_2} \right\rangle = a_1a_2 + b_1b_2 = \mathrm{Re}(z_1\overline{z_2}).$$

Set Notation

Open ball: $B(z,\delta) := \{w : |z-w| < \delta\}$. **Closed ball:** $\bar{B}(z,\delta) := \{w : |z-w| \le \delta\}$.

3. Complex Numbers Form a Field

Field Axioms

Here z_1, z_2, z_3 denote arbitrary elements of $\mathbb{C}$.
(P1) Associative addition: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.
(P2) Additive identity: $z_1 + 0 = 0 + z_1 = z_1$.
(P3) Additive inverses: $z_1 + (-z_1) = (-z_1) + z_1 = 0$.
(P4) Commutative addition: $z_1 + z_2 = z_2 + z_1$.
(P5) Associative multiplication: $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$.
(P6) Multiplicative identity: $z_1 \cdot 1 = 1 \cdot z_1 = z_1$, provided $1 \neq 0$.
(P7) Multiplicative inverses: $z_1 \cdot z_1^{-1} = z_1^{-1} \cdot z_1 = 1$ for $z_1 \neq 0$.
(P8) Commutative multiplication: $z_1 \cdot z_2 = z_2 \cdot z_1$.
(P9) Distributive law: $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$.

Division rules. $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$ (equivalently, $z\bar{z} = |z|^2$), and $\dfrac{w}{z} = \dfrac{w\bar{z}}{|z|^2}$.

4. Matrix Form of a Complex Number

For a fixed $z = a + ib$, multiplying by $w = x + iy$ is a linear transformation $M_z\vec{w} := zw$:

$$\binom{x}{y} \mapsto \binom{ax - by}{bx + ay}, \qquad M_z := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Matrix addition: $M_{z_1} + M_{z_2} = \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} + \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1+a_2 & -(b_1+b_2) \\ b_1+b_2 & a_1+a_2 \end{pmatrix} = M_{z_1+z_2}$.

Matrix multiplication: $M_{z_1} \cdot M_{z_2} = \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix}\begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 - b_1b_2 & -(a_2b_1 + a_1b_2) \\ a_2b_1 + a_1b_2 & a_1a_2 - b_1b_2 \end{pmatrix} = M_{z_1z_2}$.

Theorem (field isomorphism)

Let $\mathcal{M} := \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$. The map $\phi : \mathbb{C} \to \mathcal{M}$, $z \mapsto M_z$, is a field isomorphism – a bijection preserving both addition and multiplication.

Proof. The entries of M_z recover $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$, so ϕ is injective; it is surjective onto $\mathcal{M}$ by definition of $\mathcal{M}$. Hence ϕ is a bijection. Additivity $\phi(z_1 + z_2) = \phi(z_1) + \phi(z_2)$ and multiplicativity $\phi(z_1z_2) = \phi(z_1)\phi(z_2)$ are exactly the matrix identities $M_{z_1} + M_{z_2} = M_{z_1+z_2}$ and $M_{z_1} \cdot M_{z_2} = M_{z_1z_2}$ established above. □

The conjugate matrix. $M_z^T = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = M_{\bar{z}}$.

Note 2b — Complex Plane: Polar Form and Integer Powers

1. Polar Form of a Complex Number

A complex number z in polar form is given by its absolute value $|z|$ and an angle θ: in real coordinates, $(r\cos\theta, r\sin\theta)$; in complex notation, $z = |z|(\cos\theta + i\sin\theta)$.

Using the (temporary) notation $\mathrm{cis}\,\theta := \cos\theta + i\sin\theta$: $|\,\mathrm{cis}\,\theta| = 1$.

The argument

The angle $\theta = \arg z$ is not unique: $\arg z = \theta + 2k\pi$ for any $k \in \mathbb{Z}$. The **principal value** is restricted to $-\pi < \mathrm{Arg}\,z \le \pi$. Also $\arg\bar{z} = -\arg z = \{\theta : -\theta \in \arg z\}$. While the set $2\arg z$ is well defined independent of the chosen representative, $\dfrac{\arg z}{2}$ does depend on it.

Why the asymmetry? Fix one representative θ with $z = |z|\,\mathrm{cis}\,\theta$; every other legal choice is $\theta + 2k\pi$ for some $k \in \mathbb{Z}$, since two angles describe the same point of the plane exactly when they differ by an integer multiple of 2π. Doubling,

$$2(\theta + 2k\pi) = 2\theta + (2k)\cdot 2\pi,$$

which differs from 2θ by an integer multiple of 2π *for every* k. So $\mathrm{cis}(2\theta)$ comes out the same no matter which representative θ we started from: $2\arg z$ is well defined as a point on the unit circle, i.e. as a residue class mod 2π.

Halving is different:

$$\frac{\theta + 2k\pi}{2} = \frac{\theta}{2} + k\pi,$$

and $k\pi$ is an integer multiple of 2π only when k is *even*. When k is odd, the representative shifts by π, so

$$\mathrm{cis}\left(\frac{\theta}{2} + k\pi\right) = -\,\mathrm{cis}\left(\frac{\theta}{2}\right) \qquad (k \text{ odd}),$$

a genuinely different complex number. The representatives of $\arg z$ split into two families – one giving $\mathrm{cis}(\theta/2)$, the other $-\mathrm{cis}(\theta/2)$ – so $\dfrac{\arg z}{2}$ is only well defined mod π, not mod 2π: it does not pick out a single point of the unit circle, but a genuine two-element ambiguity. (This is exactly why $\sqrt{z}$ has two values, differing by a sign, rather than one.)

2. Rotation as Multiplication

Multiplication of complex numbers geometrically represents rotation and dilation. Rotation of $\mathbb{R}^2$ is the linear map with matrix $M_{\mathrm{cis}\,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Multiplying by $\mathrm{cis}\,\theta$ gives $\mathrm{cis}\,\theta \cdot w = M_{\mathrm{cis}\,\theta}\vec{w}$, rotating w by angle θ, so $\arg(\mathrm{cis}\,\theta \cdot w) = \arg w + \theta$. Since $z = |z|\,\mathrm{cis}(\arg z)$, the product zw is w rotated by $\arg z$ and dilated by $|z|$.

Theorem (arguments and moduli)

Multiplication: $\arg(zw) = \arg z + \arg w$, $|zw| = |z||w|$, where $\arg z + \arg w$ denotes the sum of the two residue classes modulo 2π (i.e. $\{\theta_1 + \theta_2 : \theta_1 \in \arg z,\ \theta_2 \in \arg w\}$, itself a single residue class mod 2π).
Division: $\arg(z/w) = \arg z - \arg w$, $|z/w| = |z|/|w|$ (for $w \neq 0$).

Proof. For multiplication: take $\theta_1 \in \arg z$, $\theta_2 \in \arg w$, and add. For division: $\arg(1/w) = \arg(\bar{w}/|w|^2) = -\arg w$; apply this to multiplication. □

Remark. The identity $\mathrm{Arg}(zw) = \mathrm{Arg}\,z + \mathrm{Arg}\,w$ does *not* always hold for the *principal* argument. For example, if $z = w = -i$, then $\mathrm{Arg}\,z = \mathrm{Arg}\,w = -\pi/2$, but $zw = -1$ and $\mathrm{Arg}(-1) = \pi \neq -\pi/2 + (-\pi/2)$.

3. Trigonometry and De Moivre's Formula

Trigonometry simplifies via $\mathrm{cis}(\theta_1 + \theta_2) = \mathrm{cis}\,\theta_1 \cdot \mathrm{cis}\,\theta_2$: expanding,

$$\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2),$$

and equating real/imaginary parts recovers the standard sum formulas

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2, \qquad \sin(\theta_1 + \theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2.$$

De Moivre's Formula

$$\begin{aligned} \cos(n\theta) &= \frac{1}{2}\Big((\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n\Big) \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} \sin^{2k}\theta\cos^{n-2k}\theta, \\ \sin(n\theta) &= -\frac{i}{2}\Big((\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n\Big) \\ &= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} \sin^{2k+1}\theta\cos^{n-1-2k}\theta. \end{aligned}$$

4. Powers and Roots

Integer Powers

$z^n = |z|^n\,\mathrm{cis}(n\arg z)$ for $n \in \mathbb{N}$ or $n \in \mathbb{Z}$.

Example 1. $i^{239} = 1 \cdot \mathrm{cis}(239 \cdot \pi/2) = \mathrm{cis}(3\pi/2) = -i$.

Example 2. $(1+i)^{239} = (\sqrt{2})^{239}\,\mathrm{cis}(239 \cdot \pi/4) = \sqrt{2} \cdot 2^{119} \cdot \mathrm{cis}(7\pi/4) = 2^{119}(1-i)$.

Note 2c — Complex Plane: Roots of Complex Numbers

1. Powers and Roots (continued)

Roots of Unity

Solving $z^n = 1$: $n\,\mathrm{Arg}\,z \in \{2\pi k, k \in \mathbb{Z}\}$. The solutions are the **n-th roots of unity** $1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}$, where $\omega_n := \mathrm{cis}(2\pi/n)$ is the principal root of unity.

Geometrically, these n points form a **regular n-gon inscribed in the unit circle**: multiplication by ω_n rotates the plane by $2\pi/n$, and it sends the set of roots to itself, since $\omega_n \cdot \omega_n^k = \omega_n^{k+1}$ just permutes $1, \omega_n, \ldots, \omega_n^{n-1}$ cyclically. A finite set of points on a circle that is invariant under rotation by $2\pi/n$ – and evenly spans a full turn under n such rotations – is exactly the vertex set of a regular n-gon.

n-th Roots of Complex Numbers

To find the n-th roots of $w \neq 0$ (i.e. $z^n = w$): if z_0, z_1 are two such roots, then $(z_1/z_0)^n = w/w = 1$, so z_1/z_0 is a root of unity. Hence the full set of n roots is generated from a single root z_0 as $z_0, z_0\omega_n, \ldots, z_0\omega_n^{n-1}$.

Given $w = |w|(\cos\varphi + i\sin\varphi)$, a base root is

$$z_0 = |w|^{1/n}\big(\cos(\varphi/n) + i\sin(\varphi/n)\big).$$

Example ($n = 6$, $w = 1 + i$). Here $|w| = \sqrt{2}$ and $\varphi = \pi/4$, so $z_0 = 2^{1/12}\,\mathrm{cis}(\pi/24)$, and the six roots $z_0, z_0\omega_6, \ldots, z_0\omega_6^5$ are equally spaced by angle $\pi/3$ on the circle of radius $2^{1/12}$.

Note 3 — Stereographic Projection

1. Geometric Construction

Stereographic projection maps the coordinate grid from the complex plane $\mathbb{C}$ onto the unit sphere $S \subset \mathbb{R}^3$. The sphere is centered at the origin, with $\mathbb{C}$ identified with its equatorial plane $X_3 = 0$; the projection is defined by drawing a straight line from the North Pole $N = (0, 0, 1)$ of the sphere to a point on the complex plane, and the intersection of this line with the sphere defines the corresponding point on the sphere.

The North Pole N itself has no preimage in $\mathbb{C}$ under this projection. To make the correspondence a genuine bijection, we adjoin a single **point at infinity** to $\mathbb{C}$: write $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ for the **extended complex plane**, and identify ∞ with N. Under this identification, stereographic projection becomes a bijection $\hat{\mathbb{C}} \to S$, and S (equivalently, $\hat{\mathbb{C}}$ with this geometry) is called the **Riemann sphere**.

2. Algebraic Formulas

Let $z = x + iy$, and let (X_1, X_2, X_3) be the coordinates of its projection on the unit sphere. Using similar triangles formed by the North Pole, the point on the sphere, and the point on the plane, we get an algebraic relationship between them.

Figure 1: Stereographic projection from the north pole N onto the complex plane $\mathbb{C}$ (the equatorial plane $X_3 = 0$). The point z and its spherical image $P = (X_1, X_2, X_3)$ lie on the ray through N, and F is the foot of P on the axis. The right triangles $\triangle NOZ$ and $\triangle NFP$ are similar (their horizontal legs OZ and FP are parallel), so $\dfrac{\sqrt{1 - X_3^2}}{1 - X_3} = \dfrac{|z|}{1}$, whence $|z|^2 = \dfrac{1 - X_3^2}{(1 - X_3)^2} = \dfrac{1 + X_3}{1 - X_3}$ and $X_3 = \dfrac{|z|^2 - 1}{|z|^2 + 1}$. Applying the same ratio to the in-plane components gives $X_1 = (1 - X_3)x = \dfrac{2\,\mathrm{Re}\,z}{|z|^2 + 1}$, $X_2 = (1 - X_3)y = \dfrac{2\,\mathrm{Im}\,z}{|z|^2 + 1}$, with inverse $z = \dfrac{X_1 + iX_2}{1 - X_3}$.

The projection formulas

$$X_3 = \frac{|z|^2 - 1}{|z|^2 + 1}, \qquad X_1 = (1 - X_3)x = \frac{2\,\mathrm{Re}(z)}{|z|^2 + 1}, \qquad X_2 = (1 - X_3)y = \frac{2\,\mathrm{Im}(z)}{|z|^2 + 1},$$

with inverse

$$z = \frac{X_1 + iX_2}{1 - X_3}.$$

See Figure 1 for the derivation, obtained from

$$|z|^2 = \frac{1 - X_3^2}{(1 - X_3)^2} = \frac{1 + X_3}{1 - X_3}$$

by solving for X_3 and applying the same ratio to the in-plane components.

3. Spherical Distance (Chordal Metric)

The distance between two points Z, Z' on the sphere is the Euclidean distance $d(Z, Z') := |Z - Z'|$ in $\mathbb{R}^3$. By geometric similarity and the Pythagorean theorem, the distance from the North Pole to a projected point is

$$|N - Z| = \frac{2}{\sqrt{1 + |z|^2}}.$$

Figure 2: Deriving $|N - Z|$. The right triangles $\triangle NOz$ and $\triangle NZS$ share the angle θ at N (both Z and z lie on the same ray from N), and are right-angled at O and at Z respectively – the latter by Thales' theorem, since NS is a diameter of S. Hence they are similar, so $\dfrac{NZ}{NS} = \dfrac{NO}{Nz}$, i.e. $\dfrac{|N - Z|}{2} = \dfrac{1}{\sqrt{1 + |z|^2}}$, giving $|N - Z| = \dfrac{2}{\sqrt{1 + |z|^2}}$.

Because the triangles $\triangle NZZ'$ and $\triangle Nzz'$ are similar, this yields the chordal metric.

Theorem (chordal metric)

$$d(z, z') = \frac{2|z - z'|}{\sqrt{1 + |z|^2}\sqrt{1 + |z'|^2}}, \qquad d(z, \infty) = \lim_{z' \to \infty} d(z, z') = \frac{2}{\sqrt{1 + |z|^2}}.$$

See Figure 3.

Figure 3: The chordal (spherical) metric. Points $z, z' \in \mathbb{C}$ project from the north pole N to Z, Z' on the sphere, and the chordal distance is the Euclidean chord $d(z, z') = |Z - Z'|$. Along each projection ray the product of the two distances from N is constant: $NZ \cdot Nz = NZ' \cdot Nz' = 2$, since $|N - Z| = \dfrac{2}{\sqrt{1+|z|^2}}$ and $|N - z| = \sqrt{1 + |z|^2}$. Hence $\triangle NZZ' \sim \triangle Nz'z$ (common angle at N, enclosing sides in proportion), and the ratio of corresponding sides gives $d(z, z') = \dfrac{2\,|z - z'|}{\sqrt{1 + |z|^2}\,\sqrt{1 + |z'|^2}}$. Since N corresponds to the point at infinity, letting $z' \to \infty$ slides $Z' \to N$, whence $d(z, \infty) = |N - Z| = \dfrac{2}{\sqrt{1 + |z|^2}}$.

4. Circles and Straight Lines

A fundamental property of stereographic projection is that circles and straight lines in $\mathbb{C}$ are always mapped to circles on the sphere.

Straight lines. The image of a line in the plane is the intersection of the sphere S with the plane through the line and the North Pole N – a circle passing through N.

Circles. A circle $(x-a)^2+(y-b)^2 = r^2$ in the plane, under the projection formulas, becomes the equation of a plane:

$$aX_1 + bX_2 + \frac{a^2 + b^2 - 1 - r^2}{2}X_3 = \frac{a^2 + b^2 - r^2 + 1}{2}.$$

So the image is the intersection of this plane with S. Conversely, any plane intersecting the sphere but not passing through N corresponds to a circle in $\mathbb{C}$.

Proof of the converse. Write a general plane as $AX_1+BX_2+CX_3 = D$; since it does not pass through $N = (0, 0, 1)$, we have $C \neq D$. Substituting the projection formulas and writing $z = x + iy$, $X_1 = \dfrac{2x}{|z|^2 + 1}$, $X_2 = \dfrac{2y}{|z|^2 + 1}$, $X_3 = \dfrac{|z|^2 - 1}{|z|^2 + 1}$ and clearing the denominator $|z|^2 + 1$ turns the plane equation into

$$2Ax + 2By + C(|z|^2 - 1) = D(|z|^2 + 1),$$

i.e.

$$(C - D)|z|^2 + 2Ax + 2By = C + D.$$

Since $C \neq D$, dividing by $C - D$ gives

$$x^2 + y^2 + \frac{2A}{C - D}\,x + \frac{2B}{C - D}\,y = \frac{C + D}{C - D},$$

which is the equation $(x - \alpha)^2 + (y - \beta)^2 = \rho^2$ of a circle in $\mathbb{C}$, with center $\alpha = -\dfrac{A}{C - D}$, $\beta = -\dfrac{B}{C - D}$, and $\rho^2 = \alpha^2 + \beta^2 + \dfrac{C + D}{C - D}$ (positive precisely because the plane meets the sphere in a circle, not a single tangent point). □

The excluded case. When $C = D$ (i.e. the plane passes through N), the same substitution gives $2Ax + 2By = C + D$ with no $|z|^2$ term – a *line* in $\mathbb{C}$, not a circle. This is exactly the converse of the "Straight lines" case above: a plane through N corresponds to a line in $\mathbb{C}$.

5. Transformations and Isometries

Certain transformations of $\mathbb{C}$ correspond to coordinate reflections of the sphere S and preserve the spherical distance $d(z, z')$:

- **Conjugation** $z \mapsto \bar{z}$ corresponds to $(X_1, X_2, X_3) \mapsto (X_1, -X_2, X_3)$.
  Derivation. Writing $z = x + iy$, $\bar{z} = x - iy$ has the same modulus $|z|$, so $X_3 = \dfrac{|z|^2 - 1}{|z|^2 + 1}$ is unchanged; in the formulas for X_1, X_2, replacing (x, y) by $(x, -y)$ negates only $X_2 = \dfrac{2y}{|z|^2 + 1}$.
- $z \mapsto 1/\bar{z}$ corresponds to $(X_1, X_2, X_3) \mapsto (X_1, X_2, -X_3)$.
  Derivation. $1/\bar{z} = z/|z|^2$, so $|1/\bar{z}| = 1/|z|$. Substituting $|z| \mapsto 1/|z|$ into $X_3 = \dfrac{|z|^2 - 1}{|z|^2 + 1}$ gives $\dfrac{1/|z|^2 - 1}{1/|z|^2 + 1} = \dfrac{1 - |z|^2}{1 + |z|^2} = -X_3$, while $\mathrm{Re}(1/\bar{z}) = \mathrm{Re}(z)/|z|^2$ and $|1/\bar{z}|^2 + 1 = (1 + |z|^2)/|z|^2$ combine to leave $X_1 = \dfrac{2\,\mathrm{Re}(z)}{|z|^2 + 1}$ unchanged, and likewise for X_2.
- **Inversion** $z \mapsto 1/z$ corresponds to $(X_1, X_2, X_3) \mapsto (X_1, -X_2, -X_3)$.
  Derivation. Since $1/z = \overline{1/\bar{z}}$, inversion is the composition of $z \mapsto 1/\bar{z}$ (which negates X_3) with conjugation (which negates X_2); applying both gives $(X_1, X_2, X_3) \mapsto (X_1, -X_2, -X_3)$.

All of these preserve $d(z, z')$: each corresponds to an orthogonal reflection of $\mathbb{R}^3$ (negating one or two of the coordinates X_1, X_2, X_3), and an orthogonal map preserves Euclidean distance, hence preserves $d(Z, Z') = |Z - Z'|$ by definition.

However, translations such as $z \mapsto z+1$ do *not* preserve the spherical distance: for instance $d(0,\infty)=2$ but $d(1,\infty)=\dfrac{2}{\sqrt{2}}=\sqrt{2}\neq 2$.

Note 4 — Limits and Continuity

Same as for $\mathbb{R}^2$.

1. Limits

Definition (limit)

Let f be a function defined on a set $K\subset\mathbb{C}$. We say f has a **limit** A as $z\to z_0$ if

$$\forall \varepsilon>0\ \exists \delta>0 : 0<|z-z_0|<\delta,\ z\in K \implies |f(z)-A|<\varepsilon.$$

Properties

1. If the limit exists, it is unique, provided z_0 is a limit point of K (i.e. $\forall\delta>0$: $B(z_0,\delta)\cap K\setminus\{z_0\}\neq\varnothing$).
2. $\lim_{z\to z_0}[f(z)+g(z)]=\lim_{z\to z_0}f(z)+\lim_{z\to z_0}g(z)$ (if both exist).
3. $\lim_{z\to z_0}[f(z)\cdot g(z)]=\lim_{z\to z_0}f(z)\cdot\lim_{z\to z_0}g(z)$ (if both exist).
4. $\lim_{z\to z_0}\dfrac{f(z)}{g(z)}=\dfrac{\lim_{z\to z_0}f(z)}{\lim_{z\to z_0}g(z)}$ (if both exist and $\lim_{z\to z_0}g(z)\neq 0$).
5. $\lim_{z\to z_0}f(z)=A \iff \lim_{z\to z_0}\operatorname{Re}f(z)=\operatorname{Re}A$ and $\lim_{z\to z_0}\operatorname{Im}f(z)=\operatorname{Im}A$.
6. $\lim_{z\to z_0}f(z)=A \implies \lim_{z\to z_0}\overline{f(z)}=\overline{A}$.

Proof idea.

- (1), (2), (3), (4) – as in the real case.
- (6) Follows from $|f(z)-A|=|\overline{f(z)}-\overline{A}|<\varepsilon$.
- (5) ($\Leftarrow$) $f=\operatorname{Re}f(z)+i\operatorname{Im}f(z)$; use (2). ($\Rightarrow$) $\operatorname{Re}f(z)=\frac{1}{2}\big(f(z)+\overline{f(z)}\big)$ and $\operatorname{Im}f(z)=-\frac{i}{2}\big(f(z)-\overline{f(z)}\big)$; use (6) and (2).

Important Property (restriction)

Let $K_1,K_2\subset K$, and suppose $\lim\limits_{\substack{z\to z_0\\ z\in K}}f(z)=A$. Then

$$\lim_{\substack{z\to z_0\\ z\in K_1}}f(z)=\lim_{\substack{z\to z_0\\ z\in K_2}}f(z)=A.$$

Proof. $\forall\varepsilon>0\ \exists\delta>0$: $0<|z-z_0|<\delta \implies |f(z)-A|<\varepsilon$; in particular this holds for $z\in K_1\subset K$. □

Corollary

Let $K_1,K_2\subset K$. If $\lim\limits_{\substack{z\to z_0\\ z\in K_1}}f(z)\neq\lim\limits_{\substack{z\to z_0\\ z\in K_2}}f(z)$, then $\lim\limits_{\substack{z\to z_0\\ z\in K}}f(z)$ does not exist.

Warning: the converse fails. Agreement of the limit along every *straight* path through z_0 does *not* by itself guarantee that $\lim_{z\to z_0}f(z)$ exists – a counterexample typically needs a curved path (e.g. a parabola) along which f tends to a different value.

2. Example: $\bar z/z$ Has No Limit at 0

Claim. $\lim\limits_{z\to 0}\dfrac{\bar z}{z}$ **does not exist**.

On the real axis ($z=x$): $\lim\limits_{x\to 0}\dfrac{x}{x}=1$.

On the imaginary axis ($z=iy$): $\lim\limits_{y\to 0}\dfrac{-iy}{iy}=-1$.

Alternatively, using polar coordinates $z=|z|\operatorname{cis}(\varphi)$:

$$\frac{\bar z}{z}=\frac{|z|\operatorname{cis}(-\varphi)}{|z|\operatorname{cis}(\varphi)}=\operatorname{cis}(-2\varphi).$$

This shows we get **different limits along different rays**!

3. Continuity

As usual: f is **continuous** at z_0 if $\lim\limits_{z\to z_0}f(z)=f(z_0)$.

4. Limits Involving Infinity

Definitions (limits involving ∞)

All of this can be done for the limit equal to ∞, but we need to use the **spherical metric**:

$$\lim_{z\to z_0}f(z)=\infty \iff \lim_{z\to z_0}d(f(z),\infty)=0 \iff \lim_{z\to z_0}\frac{2}{\sqrt{1+|f(z)|^2}}=0 \iff \lim_{z\to z_0}\frac{1}{|f(z)|}=0.$$

Defining the limit at infinity:

$$\lim_{z\to\infty}f(z)=A \iff \forall\varepsilon>0\ \exists\delta>0 : d(z,\infty)<\delta \implies |f(z)-A|<\varepsilon$$

$$\iff \forall\varepsilon>0\ \exists\delta>0 : \frac{1}{|z|}<\delta \implies |f(z)-A|<\varepsilon$$

$$\iff \forall\varepsilon>0\ \exists R>0 : |z|>R \implies |f(z)-A|<\varepsilon$$

$$\Big(\text{where } R=\tfrac{1}{\delta}\Big).$$

Important (and easy) observation. If $z_0\neq\infty$, then

$$\lim_{z\to z_0}|z-z_0|=0 \iff \lim_{z\to z_0}d(z,z_0)=\lim_{z\to z_0}\frac{2|z-z_0|}{\sqrt{1+|z|^2}\sqrt{1+|z_0|^2}}=0.$$

Note 5a — Analytic Functions and the Cauchy–Riemann Equations

1. Analytic Functions

Definition (complex differentiability): A function f is *(complex) differentiable* at z_0 if the limit

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

exists.

Important — where is f defined? For this to be interesting, f should be defined on a whole disc $B(z_0,r)$ for some $r>0$: unlike on the real line, the point z is free to approach z_0 from every direction of the plane, and the limit must be the same for all of them.

Equivalent definition: f is differentiable at z_0 if and only if

$$f(z)=f(z_0)+(z-z_0)\,\varphi(z),\qquad \text{where } \varphi \text{ is continuous at } z_0,$$

in which case $\varphi(z_0)=f'(z_0)$.

Proof (of equivalence). ($\Leftarrow$) If the representation holds, then $\lim\limits_{z\to z_0}\dfrac{f(z)-f(z_0)}{z-z_0}=\lim\limits_{z\to z_0}\varphi(z)=\varphi(z_0)$, so $f'(z_0)$ exists and equals $\varphi(z_0)$. ($\Rightarrow$) Conversely, take

$$\varphi(z)=\begin{cases}\dfrac{f(z)-f(z_0)}{z-z_0}, & z\neq z_0,\\[2mm] f'(z_0), & z=z_0;\end{cases}$$

continuity of φ at z_0 is precisely the statement that f is differentiable there. □

Analytic (holomorphic) functions

f is called **analytic** (or **holomorphic**) at a point $z_0\in\mathbb{C}$ if for some $\delta>0$ it has a derivative at *every* point $z\in B(z_0,\delta)$. A function analytic at every point of an open set is analytic there; a function analytic on all of $\mathbb{C}$ is called **entire**.

Remark — one point is not enough. Differentiability at a single point (or even along a curve) is *not* interesting. For instance, $F(z)=\operatorname{Re}z\,|\operatorname{Re}z|$ is complex differentiable at every point of the line $\operatorname{Re}z=0$ and nowhere else; in particular it is analytic *nowhere*. What matters is differentiability at every point of a whole neighborhood.

Arithmetic (the same as in Calculus). If $f'(z)$ and $g'(z)$ exist, then $(f\pm g)'=f'\pm g'$, $(fg)'=f'g+fg'$, and $\Big(\frac{f}{g}\Big)'=\frac{f'g-fg'}{g^2}$ where $g(z)\neq 0$. If $f'(z)$ exists and g' exists at $f(z)$, then $\Big(g(f(z))\Big)'=g'(f(z))\cdot f'(z)$ (Chain Rule). The proofs are identical to the real case.

2. Real vs. Complex Differentiability

Identifying $\mathbb{C}$ with $\mathbb{R}^2$ through $z=x+iy$, a function $f=u+iv$ becomes a map $\mathbb{R}^2\to\mathbb{R}^2$. Recall that f is **real-differentiable** at z_0 if there is a real-linear map T with

$$f(z_0+h)=f(z_0)+T(h)+o(|h|),\qquad h\to 0.$$

By contrast, **complex differentiability** demands the sharper estimate

$$\lim_{h\to 0}\frac{\big|f(z_0+h)-f(z_0)-f'(z_0)\,h\big|}{|h|}=0,$$

i.e. the increment is approximated not merely by a real-linear map but by *multiplication by the complex number* $f'(z_0)$.

From calculus, the differential of $f=u+iv$ is

$$T(h)=\frac{\partial f}{\partial x}\,\xi+\frac{\partial f}{\partial y}\,\eta=\begin{pmatrix}u_x & u_y\\ v_x & v_y\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\qquad\text{(real form)},\quad h=\xi+i\eta.$$

Substituting $\xi=\frac{h+\bar h}{2}$ and $\eta=\frac{h-\bar h}{2i}$ and collecting the coefficients of h and $\bar h$ gives the complex form

$$T(h)=\frac{\partial f}{\partial z}(z_0)\,h+\frac{\partial f}{\partial\bar z}(z_0)\,\bar h,$$

where the coefficients are the **Wirtinger derivatives**

$$\frac{\partial f}{\partial z}=\frac{1}{2}\Big(\frac{\partial f}{\partial x}-i\,\frac{\partial f}{\partial y}\Big),\qquad \frac{\partial f}{\partial\bar z}=\frac{1}{2}\Big(\frac{\partial f}{\partial x}+i\,\frac{\partial f}{\partial y}\Big).$$

3. The Cauchy–Riemann Equations

Dividing the real-differentiable expansion by h and letting $h\to 0$,

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}=\lim_{h\to 0}\frac{T(h)}{h}=\frac{\partial f}{\partial z}(z_0)+\frac{\partial f}{\partial\bar z}(z_0)\cdot\lim_{h\to 0}\frac{\bar h}{h}.$$

The last limit does *not* exist ($\bar{h}/h = 1$ along the real axis, -1 along the imaginary axis), so the whole limit exists precisely when the offending term vanishes. This proves:

Theorem (Cauchy–Riemann)

Let f be a real-differentiable function at z_0. Then f is complex differentiable at z_0 **if and only if**

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0.$$

In that case $f'(z_0) = \dfrac{\partial f}{\partial z}(z_0)$.

Warning: real-differentiability is a standing assumption. The equivalence above is only as good as its hypothesis: it presupposes that f is already real-differentiable at z_0. Dropping that hypothesis and keeping only the Cauchy–Riemann equations is *not* enough to conclude analyticity in general – this subtlety (and the Looman–Menchoff theorem) is discussed in a later note.

Writing $f = u + iv$ and expanding $\partial f/\partial \bar{z} = 0$ via the Wirtinger formula,

$$\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y}\right) = 0,$$

and separating real and imaginary parts yields the classical system.

The Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Matrix interpretation. The Cauchy–Riemann equations say exactly that the Jacobian of f has the special form

$$\begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ -\dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial x} \end{pmatrix} = M_{f'(z_0)},$$

the matrix of multiplication by the complex number $f'(z_0)$. In other words, complex differentiability means the differential — a priori only real-linear — is in fact complex-linear. Consequently the derivative can be read off from either component alone:

$$f'(z) = \frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\,\frac{\partial v}{\partial x}.$$

4. A Real Analysis Refresher

A common trap is to believe that the mere *existence* of the partial derivatives is enough. Before reading on, try the quiz.

Question

Suppose $f = u + iv : \mathbb{R}^2 \to \mathbb{R}^2$ has all first partial derivatives at every point $z \in B(z_0, r)$. What can you say about f?

1. f is continuous in $B(z_0, r)$, but not always differentiable.
2. f is differentiable in $B(z_0, r)$.
3. The functions u and v are differentiable in $B(z_0, r)$.
4. f can be discontinuous at some points of $B(z_0, r)$.

The correct answer is **(4)**: the existence of partials does not even force continuity.

Counterexample. With $z = x + iy$, let

$$f(z) = \begin{cases} \dfrac{xy}{x^2+y^2} + i \cdot 0, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

On both coordinate axes $f \equiv 0$, so all four partials vanish at the origin,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}\bigg|_0 = 0,$$

and the Cauchy–Riemann equations hold at $z = 0$. Yet f is **not continuous** at 0: along the diagonal $z = t(1+i)$,

$$f(z) = \frac{t^2}{2t^2} = \frac{1}{2} \not\to 0.$$

What rescues the situation is *continuity of the partials*.

Note 5b — When Do the CR Equations Imply Analyticity?

1. A Real Analysis Refresher (continued)

Theorem (Multivariable Calculus)

Let $u(x, y)$ have partial derivatives $\dfrac{\partial u}{\partial x}(x, y)$ and $\dfrac{\partial u}{\partial y}(x, y)$ for all $(x, y) \in B\big((x_0, y_0), \delta\big)$ which are **continuous** at (x_0, y_0). Then u is differentiable at (x_0, y_0).

Working criterion. Combining this with the Cauchy–Riemann theorem: if the partials of u and v exist and are continuous on $B(z_0, r)$ and satisfy the Cauchy–Riemann equations there, then $f = u + iv$ is analytic in $B(z_0, r)$.

2. Do the CR Equations Alone Imply Analyticity?

The working criterion above assumes real-differentiability (secured by continuous partials). Can we drop it? Not naively.

Remark — some regularity must be assumed a priori. There exists f whose Wirtinger derivatives $\dfrac{\partial f}{\partial z}, \dfrac{\partial f}{\partial \bar{z}}$ exist everywhere, with $\dfrac{\partial f}{\partial \bar{z}} \equiv 0$, yet f is *not* everywhere analytic. Looman's example is

$$f(z) = \begin{cases} e^{-1/z^4}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

(This is the same "flat function" trick behind the standard real-analysis example e^{-1/x^2}, smooth but not real-analytic at 0.) Away from the origin f is a composition of analytic functions, so $\partial f/\partial \bar{z} = 0$ there. At the origin the partials exist and vanish (along both axes $z^4 > 0$, so $f \to 0$ smoothly), hence $\partial f/\partial \bar{z}(0) = 0$ as well. Nevertheless f is not even continuous at 0: along the ray $\arg z = \pi/4$ we have $z^4 = -r^4$, so $|f(z)| = e^{1/r^4} \to \infty$. The point is that the partials fail to be *continuous* at 0 — so the working criterion does not apply, and there is no contradiction.

Continuity of f itself, however, *is* enough — a deep and surprising fact.

Theorem (Looman–Menchoff)

Let $f = u + iv$ be **continuous** on $B(z_0, r)$, suppose all the partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ **exist** on $B(z_0, r)$, and suppose the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

hold there. Then f is analytic in $B(z_0, r)$.

No continuity of the partials, and no a priori differentiability, is required — only continuity of f, existence of the partials, and the Cauchy–Riemann equations. (Due to H. Looman, 1923, and D. Menshov, 1936; it strengthens an earlier theorem of Goursat.)

3. Vanishing Derivative and Constants

Theorem

Let f be analytic in $B(z_0, \delta)$ with $f'(z) \equiv 0$ for all $z \in B(z_0, \delta)$. Then $f(z) \equiv \text{const}$.

Proof. From $f'(z) = 0$,

$$\frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\,\frac{\partial v}{\partial x} \equiv 0 \implies \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} \equiv 0.$$

By calculus, a function on a (connected) disc with vanishing gradient is constant, so $u \equiv \text{const}$ and $v \equiv \text{const}$. ∎

Remark — one can assume less. It is enough to require

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \equiv 0$$

without assuming differentiability a priori: the differentiability follows from continuity.

Proof. Since $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ vanish identically they are continuous, so by the calculus theorem f is real-differentiable. Then

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\,\frac{\partial f}{\partial y}\right) = 0 \implies f \text{ is analytic,}$$

while $\dfrac{\partial f}{\partial z} = \frac{1}{2}\big(\partial_x - i\,\partial_y\big) f = 0 = f'$, hence f is constant. ∎

4. Examples

- **The exponential is entire.** For $f(z) = e^x(\cos y + i \sin y)$ we have $u = e^x \cos y$, $v = e^x \sin y$, with continuous partials satisfying
$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}.$$
Hence e^z is entire, and $f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = e^z$.
- **Conjugation is nowhere differentiable.** For $f(z) = \bar{z} = x - iy$ we get $u_x = 1$ but $v_y = -1$, so the first CR equation fails everywhere; equivalently $\partial f/\partial \bar{z} = 1 \neq 0$. Thus $\bar{z}$ is differentiable nowhere.
- **Differentiable at one point only.** For $f(z) = |z|^2 = z\bar{z}$ we have $u = x^2 + y^2$, $v = 0$. The CR equations read $2x = 0$ and $2y = 0$, satisfied only at the origin. So f is complex differentiable at $z = 0$ (with $f'(0) = 0$) and nowhere else — analytic at no point of the plane. Differentiability at a point does not propagate to a neighborhood.

Note 6 — Harmonic Functions

1. From Analytic to Harmonic

We identify a point $(x, y) \in \mathbb{R}^2$ with the complex number $z = x + iy \in \mathbb{C}$, and write $B(z_0, r)$ for the open disc of radius r about z_0. Our aim is to

isolate the single second-order equation satisfied by the real and imaginary parts of an analytic function.

Reminder — the classes C^2 and C^∞. A real function g on an open set $\Omega \subseteq \mathbb{R}^2$ is of class C^k if all of its partial derivatives up to order k exist and are continuous on Ω. Thus $g \in C^2$ means that g and the derivatives $g_x, g_y, g_{xx}, g_{xy}, g_{yx}, g_{yy}$ are all continuous, while $g \in C^\infty$ means $g \in C^k$ for every k — all partial derivatives of every order exist and are continuous. For $g \in C^2$ the mixed partials agree, $g_{xy} = g_{yx}$ (Clairaut–Schwarz); this is the one calculus fact we lean on below.

Our starting point is a fact whose proof is deferred; it will follow later from the Cauchy integral formula.

Theorem (stated now, proved later)

Every function analytic on a disc $B(z_0, r)$ is infinitely complex differentiable there. In particular, its real and imaginary parts, viewed as functions of (x, y), are of class C^∞ on $B(z_0, r)$.

The point we extract is that these parts satisfy a single second-order equation.

Corollary (real and imaginary parts)

Let $f = u + iv$ be analytic on $B(z_0, r)$, with $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ real-valued. Then

$$u_{xx} + u_{yy} = 0 \qquad \text{and} \qquad v_{xx} + v_{yy} = 0 \qquad \text{on } B(z_0, r).$$

Proof. By the theorem above $u, v \in C^\infty$, so all second partials exist and the mixed partials commute. Analyticity is equivalent to the Cauchy–Riemann equations

$$u_x = v_y, \qquad u_y = -v_x.$$

Differentiating the first with respect to x and the second with respect to y,

$$u_{xx} = v_{yx}, \qquad u_{yy} = -v_{xy},$$

and adding, using $v_{yx} = v_{xy}$,

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.$$

The computation for v is identical: differentiating the first equation with respect to y and the second with respect to x gives $v_{yy} = u_{xy}$ and $v_{xx} = -u_{yx}$, whence $v_{xx} + v_{yy} = 0$. □

2. The Laplace Operator

The operator appearing on the left-hand side above deserves a name.

The Laplace operator

For $u \in C^2(\Omega)$ the **Laplace operator** (or **Laplacian**) is

$$\Delta u \;:=\; \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \;=\; u_{xx} + u_{yy}.$$

Recall the **Wirtinger derivatives** from the previous section,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right).$$

They factor the Laplacian.

Proposition (factorization of the Laplacian)

On $C^2(\Omega)$,

$$\Delta \;=\; 4\,\frac{\partial}{\partial z}\,\frac{\partial}{\partial \bar z} \;=\; 4\,\frac{\partial}{\partial \bar z}\,\frac{\partial}{\partial z}.$$

Proof. Using the Wirtinger formulas,

$$4\,\frac{\partial}{\partial z}\,\frac{\partial}{\partial \bar z} = 4\cdot\tfrac{1}{2}\big(\partial_x - i\,\partial_y\big)\cdot\tfrac{1}{2}\big(\partial_x + i\,\partial_y\big) = \big(\partial_x - i\,\partial_y\big)\big(\partial_x + i\,\partial_y\big) = \partial_x^2 + \partial_y^2 + i\big(\partial_x\partial_y - \partial_y\partial_x\big).$$

On C^2 functions the mixed partials cancel, leaving $\partial_x^2 + \partial_y^2 = \Delta$; reversing the order of the factors gives $4\,\partial_{\bar z}\partial_z$ in the same way. □

This gives a one-line proof of the Corollary. If f is analytic then $\partial f/\partial \bar z = 0$ (the Cauchy–Riemann theorem from the previous lecture's notes), so

$$\Delta f = 4\,\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \bar z}\right) = 0;$$

and since Δ has real coefficients, $\Delta u = \operatorname{Re}(\Delta f) = 0$ and $\Delta v = \operatorname{Im}(\Delta f) = 0$.

3. Harmonic Functions: Definition and Examples

Definition (harmonic function)

Let $\Omega \subseteq \mathbb{C}$ be open. A function $u \in C^2(\Omega)$ is called **harmonic** on Ω if

$$\Delta u = 0 \qquad \text{everywhere on } \Omega.$$

Unless stated otherwise u is real-valued; a complex-valued u is harmonic if and only if $\operatorname{Re} u$ and $\operatorname{Im} u$ are.

In this language the Corollary reads: *the real and imaginary parts of an analytic function are harmonic.* The examples below are precisely the real and imaginary parts of the simplest analytic functions.

- $u = x^2 - y^2 = \operatorname{Re}(z^2)$ **on** $\mathbb{C}$**.** Here $u_{xx} = 2$ and $u_{yy} = -2$, so $\Delta u = 0$.
- $u = xy = \frac{1}{2}\operatorname{Im}(z^2)$ **on** $\mathbb{C}$**.** Here $u_{xx} = u_{yy} = 0$, so $\Delta u = 0$ trivially.
- $u = e^x \cos y = \operatorname{Re}(e^z)$ **on** $\mathbb{C}$**.** We compute $u_{xx} = e^x \cos y$ and $u_{yy} = -e^x \cos y$, so $\Delta u = 0$.
- $u = \log|z| = \operatorname{Re}(\log z)$ **on** $\mathbb{C} \setminus \{0\}$**.** A direct computation gives $\Delta u = 0$ on the punctured plane. This example, harmonic away from the origin, will reappear later.

4. The Local Converse

The Corollary showed that the real part of an analytic function is harmonic. The converse holds, at least locally.

Theorem (local converse)

Let u be real-valued and harmonic on a disc $B(z_0, r)$. Then there exists a function f analytic on $B(z_0, r)$ such that

$$u = \operatorname{Re} f.$$

The function f is unique up to an additive imaginary constant.

The proof requires contour integration and is therefore postponed; once integrals are available we will establish a more general version, valid on any simply connected domain.

Definition (harmonic conjugate)

If u is harmonic on Ω, a real function v for which $u + iv$ is analytic on Ω is called a **harmonic conjugate** of u.

Note 7a — Polynomial Functions

1. Polynomials Are Entire

A **polynomial function** is

$$f(z) = \sum_{n=0}^{d} a_n z^n.$$

By the differentiation rules for products and sums, $f'(z) = \sum_{n=1}^{d} n a_n z^{n-1}$, so f is **entire**: analytic at every $z \in \mathbb{C}$.

Example. $p(z) = z^2 = (x^2 - y^2) + 2ixy$, where $x = \dfrac{z + \bar z}{2}$, $y = \dfrac{z - \bar z}{2i}$ – the real and imaginary parts of z^2 recovered from its expression in $z, \bar z$.

Proposition (which polynomials in x, y are analytic)

Let $F(x, y) = \sum_{j,k \ge 0} c_{jk}\, x^j y^k$ be a polynomial in the real variables x, y with complex coefficients, and define $f(z) := F(x, y)$ for $z = x + iy$. Then f is analytic on $\mathbb{C}$ if and only if, after substituting $x = \frac{z+\bar z}{2}$, $y = \frac{z-\bar z}{2i}$ and expanding, f involves no $\bar z$ – i.e. $f(z)$ is itself a polynomial in z alone.

Proof. ($\Leftarrow$) If $f(z) = p(z)$ is a polynomial in z alone, it is entire by the computation above.

($\Rightarrow$) The substitution $x = \frac{z+\bar z}{2}$, $y = \frac{z-\bar z}{2i}$ is an invertible linear change of coordinates, so expanding F produces a unique representation $f(z) = \sum_{m,n \ge 0} a_{mn}\, z^m \bar z^n$. Since F is a real polynomial, f is automatically real-differentiable everywhere; if f is also analytic, the Cauchy–Riemann criterion $\partial f/\partial \bar z \equiv 0$ gives

$$0 \equiv \frac{\partial f}{\partial \bar z} = \sum_{m \ge 0,\, n \ge 1} n\, a_{mn}\, z^m \bar z^{n-1},$$

and since the representation in $z, \bar z$ is unique, every coefficient a_{mn} with $n \ge 1$ must vanish. Hence $f(z) = \sum_m a_{m0} z^m$ is a polynomial in z alone. □

2. Roots and Multiplicity

Fundamental Theorem of Algebra (stated; proof deferred)

If p is a polynomial of degree $d \ge 1$, then p has a root (zero) z_1: $p(z_1) = 0$.

Dividing p by $(z - z_1)$: $p(z) = q(z)(z - z_1)$, where the remainder is the constant $p(z_1) = 0$. The quotient q has degree $d - 1$, so it too has a root, and by induction

$$p(z) = a_d (z - z_1) \cdots (z - z_d).$$

Some of the roots $z_1, \dots, z_d$ may coincide.

Definition (multiplicity)

The **multiplicity** (or **order**) of z_0 as a root of p is $\#\{j : z_j = z_0\}$. Equivalently, $p(z) = (z - z_0)^h q(z)$ with $q(z_0) \ne 0$: h is the order. This is equivalent to

$$p(z_0) = p'(z_0) = \cdots = p^{(h-1)}(z_0) = 0, \qquad p^{(h)}(z_0) \ne 0.$$

Lemma. If $p(z) = (z - z_0) g(z)$ for a polynomial g, then $p^{(k)}(z_0) = k\, g^{(k-1)}(z_0)$ for every $k \ge 1$.

Proof. Induction on k. For $k = 1$, the product rule gives $p'(z) = g(z) + (z - z_0) g'(z)$, so $p'(z_0) = g(z_0)$. Assuming $p^{(k)}(z) = k\, g^{(k-1)}(z) + (z - z_0) g^{(k)}(z)$, differentiating once more gives

$$p^{(k+1)}(z) = k\, g^{(k)}(z) + g^{(k)}(z) + (z - z_0) g^{(k+1)}(z) = (k+1)\, g^{(k)}(z) + (z - z_0) g^{(k+1)}(z),$$

the same identity with $k + 1$ in place of k. Evaluating at z_0 at each step gives $p^{(k)}(z_0) = k\, g^{(k-1)}(z_0)$. □

Proof of the equivalence, by induction on h. For $h = 0$, both sides say

$p(z_0) \neq 0$ (taking $q = p$ on the left; the right side is $p^{(0)}(z_0) = p(z_0) \neq 0$, with no vanishing conditions), so the two are trivially equivalent.

Assume the equivalence holds for h; we prove it for $h+1$.

($\Rightarrow$) Suppose $p(z) = (z-z_0)^{h+1}q(z)$, $q(z_0) \neq 0$. Write $p(z) = (z-z_0)g(z)$ with $g(z) := (z-z_0)^h q(z)$; since g itself satisfies the order-h hypothesis, the inductive hypothesis gives $g(z_0) = \cdots = g^{(h-1)}(z_0) = 0$ and $g^{(h)}(z_0) \neq 0$. By the Lemma, $p^{(k)}(z_0) = k\,g^{(k-1)}(z_0)$ for $k \geq 1$ (and $p(z_0) = 0$ directly, since p has the factor $z - z_0$); so $p^{(k)}(z_0) = 0$ for $1 \leq k \leq h$, while $p^{(h+1)}(z_0) = (h+1)\,g^{(h)}(z_0) \neq 0$. Together with $p(z_0) = 0$, this is exactly the order-$(h+1)$ derivative condition.

($\Leftarrow$) Suppose $p(z_0) = \cdots = p^{(h)}(z_0) = 0$ and $p^{(h+1)}(z_0) \neq 0$. Since $p(z_0) = 0$, $p(z) = (z-z_0)g(z)$ for the polynomial $g(z) := p(z)/(z-z_0)$. By the Lemma, $0 = p^{(k)}(z_0) = k\,g^{(k-1)}(z_0)$ for $1 \leq k \leq h$, so $g(z_0) = \cdots = g^{(h-1)}(z_0) = 0$; and $0 \neq p^{(h+1)}(z_0) = (h+1)\,g^{(h)}(z_0)$, so $g^{(h)}(z_0) \neq 0$. By the inductive hypothesis applied to g, this means $g(z) = (z-z_0)^h q(z)$ with $q(z_0) \neq 0$, hence $p(z) = (z-z_0)^{h+1}q(z)$, $q(z_0) \neq 0$. □

Proposition

If z_0 is a root of p of order $h \geq 1$, then z_0 is a root of p' of order $h-1$.

Proof. Write $p(z) = (z-z_0)^h q(z)$, $q(z_0) \neq 0$. Then

$$p'(z) = h(z-z_0)^{h-1}q(z) + (z-z_0)^h q'(z) = (z-z_0)^{h-1}\underbrace{\big[h\,q(z) + (z-z_0)q'(z)\big]}_{=:r(z)},$$

and $r(z_0) = h\,q(z_0) \neq 0$. So z_0 has order $h-1$ as a root of p'. □

3. The Gauss–Lucas Theorem

Theorem (Gauss–Lucas)

Let $z_1, \ldots, z_J$ be the (distinct) zeros of a polynomial p. Then all zeros of $p'(z)$ lie inside the convex hull of $\{z_1, \ldots, z_J\}$.

Definition (convex hull)

The **convex hull** of $\{z_1, \ldots, z_J\}$ is

$$H = \left\{ w \in \mathbb{C} : \exists\, m_1, \ldots, m_J \geq 0,\ \sum_j m_j = 1,\ w = \sum_j m_j z_j \right\},$$

geometrically, the smallest convex set containing $\{z_1, \ldots, z_J\}$.

Corollary. For every $c \in \mathbb{C}$, the roots of $p'(z)$ lie in the convex hull of the roots of $p(z) = c$, since $(p(z) - c)' = p'(z)$.

4. Taylor Polynomials

Theorem (Taylor polynomial)

Let p be a polynomial of degree d and $z_0 \in \mathbb{C}$. Then

$$p(z) = \sum_{k=0}^{d} \frac{p^{(k)}(z_0)}{k!}(z-z_0)^k.$$

Proof. Induction on d. For $d = 0$, $p(z) = c = p(z_0)$, and the formula holds trivially.

Inductive step. Since $p(z) - p(z_0)$ vanishes at z_0, it is divisible by $(z - z_0)$:

$$q(z) := \frac{p(z) - p(z_0)}{z - z_0}, \qquad p(z) = p(z_0) + (z-z_0)q(z),$$

where q is a polynomial of degree $d-1$. By the Lemma above, $p^{(k)}(z_0) = k\,q^{(k-1)}(z_0)$ for $k \geq 1$, i.e., $q^{(j)}(z_0) = \dfrac{p^{(j+1)}(z_0)}{j+1}$ for $j \geq 0$.

By the inductive hypothesis applied to q (degree $d-1$),

$$q(z) = \sum_{k=0}^{d-1} \frac{q^{(k)}(z_0)}{k!}(z-z_0)^k = \sum_{k=0}^{d-1} \frac{p^{(k+1)}(z_0)}{(k+1)!}(z-z_0)^k.$$

Therefore

$$p(z) = p(z_0) + (z-z_0)\sum_{k=0}^{d-1} \frac{p^{(k+1)}(z_0)}{(k+1)!}(z-z_0)^k = p(z_0) + \sum_{k=1}^{d} \frac{p^{(k)}(z_0)}{k!}(z-z_0)^k = \sum_{k=0}^{d} \frac{p^{(k)}(z_0)}{k!}(z-z_0)^k.$$

This completes the induction. □

Note 7b — The Gauss–Lucas Theorem

Theorem (Gauss–Lucas)

Let $z_1, \ldots, z_J$ be the (distinct) zeros of a polynomial p. Then all zeros of $p'(z)$ lie inside the convex hull of $\{z_1, \ldots, z_J\}$.

Proof of the Theorem. Writing $p(z) = a_d \prod_{j=1}^{J}(z-z_j)^{m_j}$, logarithmic differentiation gives

$$\frac{p'(z)}{p(z)} = \sum_{j=1}^{J} \frac{m_j}{z - z_j} = \sum_j m_j \frac{\overline{z - z_j}}{|z - z_j|^2}.$$

If z_0 is itself a multiple root of p, there is nothing to prove: $z_0 \in \{z_1, \ldots, z_J\} \subset H$. (By the Proposition established when discussing roots and multiplicity (in the companion note), a common root of p and p' is automatically a multiple root of p – a simple root of p has order 0, i.e. is not a root at all, in p' – so this covers every root z_0 of p' with $p(z_0) = 0$.) Otherwise, suppose z_0 is not a root of p, but is a root of p'. Then $p'(z_0)/p(z_0) = 0$, i.e.,

$$\sum_j m_j \frac{\overline{z_0 - z_j}}{|z_0 - z_j|^2} = 0, \qquad \text{so} \qquad \bar{z}_0 = \sum_j \mu_j \bar{z}_j, \qquad \mu_j := \frac{m_j/|z_0 - z_j|^2}{\sum_i m_i/|z_0 - z_i|^2}.$$

Conjugating, $z_0 = \sum_j \mu_j z_j$ with $\mu_j \geq 0$ and $\sum_j \mu_j = 1$. So $z_0 \in H$. □

Note 8a — Rational Functions

1. Poles and Zeros

Definition (rational function)

$R(z) = \dfrac{P(z)}{Q(z)}$, where P, Q are polynomials with no common zeros (any common zeros can be factored out and cancelled).

$$R'(z) = \frac{P'(z)Q(z) - Q'(z)P(z)}{Q(z)^2}$$

exists wherever z is not a pole.

Definition (zeros of R)

The zeros of $P(z)$ are called the **zeros** of $R(z)$. The order (multiplicity) of a zero equals the order of the corresponding zero of $P(z)$.

Definition (poles of R)

The zeros of $Q(z)$ are called the **poles** of $R(z)$. The order (multiplicity) of a pole equals the order of the corresponding zero of $Q(z)$.

Example. $R(z) = \dfrac{(z-i)^2(z+1)}{(z+i)^3}$ has a pole at $-i$ of order 3, a zero at -1 of order 1, and a zero at i of order 2.

Remark. If z_0 is a pole of R, then $\lim_{z \to z_0} R(z) = \infty$ (in the spherical metric), so we set $R(z_0) := \infty$.

2. Behavior at Infinity

Consider $R_1(z) := R(1/z) = \dfrac{P(1/z)}{Q(1/z)}$ (the "$1/z$ trick"), and define

$$R(\infty) := R_1(0) = \lim_{z \to \infty} R(z) = \lim_{z \to 0} R_1(z).$$

More details. If $P(z) = \sum_{k=0}^{n} a_k z^k$ and $Q(z) = \sum_{k=0}^{m} \ell_k z^k$ ($n = \deg P$, $m = \deg Q$, $a_n, \ell_m \neq 0$), then

$$R_1(z) = z^{m-n}\,\frac{\sum_{k=0}^{n} a_k z^{n-k}}{\sum_{k=0}^{m} \ell_k z^{m-k}} =: z^{m-n} R_0(z), \qquad R_0(0) = \frac{a_n}{\ell_m} \neq 0, \infty.$$

So

$$R(\infty) = R_1(0) = \begin{cases} \dfrac{a_n}{\ell_m}, & \deg P = \deg Q \\ 0, & \deg P < \deg Q \\ \infty, & \deg P > \deg Q. \end{cases}$$

- If $\deg P < \deg Q$: ∞ is a zero of $R(z)$, of order $\deg Q - \deg P$ (the order of 0 as a zero of $R_1(z)$, since $R_1(z) = z^{m-n}R_0(z)$ with $R_0(0) \neq 0$).
- If $\deg P > \deg Q$: ∞ is a pole of $R(z)$, of order $\deg P - \deg Q$.
- If $\deg P = \deg Q$: ∞ is neither a pole nor a zero.

Total Count of Zeros and Poles

Suppose $\deg P > \deg Q$. The total number of zeros, counting order, is

$$\#\{\text{zeros of } P\} + \#\{\text{zeros at } \infty\} = \deg P + 0 = \deg P,$$

since ∞ is a pole, not a zero, in this case; and the total number of poles is

$$\#\{\text{zeros of } Q\} + \#\{\text{poles at } \infty\} = \deg Q + (\deg P - \deg Q) = \deg P.$$

Both totals equal $\deg P = \max(\deg P, \deg Q)$.

The case $\deg P < \deg Q$ follows by the same computation applied to $1/R = Q/P$, which swaps the roles of P, Q (and hence of zeros and poles): both totals equal $\deg Q = \max(\deg P, \deg Q)$. If $\deg P = \deg Q =: d$, then ∞ is neither a zero nor a pole, so both totals reduce to the finite count alone – $\#\{\text{zeros of } P\} = d$ and $\#\{\text{zeros of } Q\} = d$ – again equal to $\max(\deg P, \deg Q) = d$.

Consequence

The total number of zeros of R equals the total number of poles of R, equal to $\max(\deg P, \deg Q)$.

Definition (order of a rational function)

$\max(\deg P, \deg Q)$ is called the **order** of $R(z)$.

Remark. For every $w \in \mathbb{C}$ with $w \neq R(\infty)$, the equation $R(z) = w$ has $\deg R$ *finite* roots, counting multiplicity, since $R(z) - w$ has the same poles as $R(z)$ (as w is finite), hence the same order, and by the Consequence above its zeros – the solutions of $R(z) = w$ – number $\deg R$ in total. (If $w = R(\infty)$, one of these $\deg R$ roots, counted on the Riemann sphere, is $z = \infty$ itself, so the number of *finite* solutions is then $\deg R$ minus the

multiplicity of ∞ as a root.)

Example 1**:** the Möbius (linear fractional) maps, $\dfrac{az+b}{cz+d}$, $ad-bc\neq 0$, $a,b,c,d\in\mathbb{C}$. Polynomials are exactly the rational functions with all poles at ∞.

3. Partial Fraction Decomposition

Singular Part at Infinity

If $R(z)=P(z)/Q(z)$ with $\deg P>\deg Q$, polynomial division gives

$$P(z)=G_\infty(z)Q(z)+S(z),\qquad \deg S<\deg Q,$$

so

$$R(z)=G_\infty(z)+\frac{S(z)}{Q(z)},\qquad H(z):=\frac{S(z)}{Q(z)},\quad H(\infty)=0.$$

G_∞ is a polynomial with $\deg G_\infty=\deg P-\deg Q>0$ and $G_\infty(\infty)=\infty$: this is the **singular part of R at ∞**. If $\deg P<\deg Q$, set $G_\infty(z)\equiv 0$; if $\deg P=\deg Q$, set $G_\infty(z)\equiv R(\infty)$, a constant.

Observation. $\deg G_\infty=\max(\deg P-\deg Q,\ 0)=$ order of ∞ as a pole of R.

Note 8b — Rational Functions: Partial Fraction Decomposition

1. Partial Fraction Decomposition (continued)

Singular Part at a Finite Pole

Let z_0 be a (finite) pole of R. Consider

$$R_1(\zeta):=R\Big(z_0+\frac{1}{\zeta}\Big),$$

a rational function of ζ with $R_1(\infty)=R(z_0)=\infty$, so (as in the previous subsection) $R_1(\zeta)=P(\zeta)+H(\zeta)$ for some polynomial P with $H(\infty)=0$. Unlike the singular part G_∞ of R itself, the constant term $P(0)$ here belongs to the *regular* behavior of R at z_0, not to its principal part (a Laurent principal part at a *finite* point never has a constant term) – so we discard it:

$$G_{z_0}(\zeta):=P(\zeta)-P(0),\qquad H_{z_0}(\zeta):=R_1(\zeta)-G_{z_0}(\zeta)=H(\zeta)+P(0),$$

so that $G_{z_0}(0)=0$ and $H_{z_0}(\infty)=P(0)$.

Changing back via $\zeta=1/(z-z_0)$, i.e. $R_1\big(1/(z-z_0)\big)=R(z)$, we get

$$R(z)-G_{z_0}\left(\frac{1}{z-z_0}\right)=H_{z_0}\left(\frac{1}{z-z_0}\right),$$

which is *regular* (finite, not a pole) at z_0 – it tends to $H_{z_0}(\infty)=P(0)$ as $z\to z_0$. The term $G_{z_0}\big(1/(z-z_0)\big)$ is a polynomial in $1/(z-z_0)$ with *no constant term*: among finite points it only has a pole at z_0, it vanishes as $z\to\infty$ (crucial for the assembly below), and $\deg G_{z_0}$ is exactly the order of z_0 as a pole of R.

Assembling the Decomposition

Let $z_1,\dots,z_h$ be all the finite poles of R, and set

$$\Psi(z):=R(z)-G_\infty(z)-\sum_{k=1}^{h}G_{z_k}\left(\frac{1}{z-z_k}\right).$$

Each $G_{z_k}(1/(z-z_k))$ is a polynomial in $1/(z-z_k)$, so (by the previous subsection) it cancels the pole of R at z_k without introducing a pole anywhere else; combined with G_∞ cancelling the behavior at ∞, Ψ has no poles anywhere on the Riemann sphere. In particular Ψ has no finite poles, so it is a polynomial; and having no pole at ∞ either forces (by the classification of behavior at infinity discussed in the companion note) that polynomial to have degree 0 – i.e. $\Psi\equiv c$ for some constant c, with no need to invoke boundedness of a general entire function. To find c, let $z\to\infty$: $R(z)-G_\infty(z)\to 0$ by construction, and *each* $G_{z_k}(1/(z-z_k))\to G_{z_k}(0)=0$, since G_{z_k} was built with no constant term. Hence $\Psi(z)\to 0$, so $c=0$. Therefore:

Partial Fraction Decomposition

$$R(z)=G_\infty(z)+\sum_{k=1}^{h}G_{z_k}\left(\frac{1}{z-z_k}\right),$$

where G_∞ is the singular part of R at ∞ and each G_{z_k} is the singular part of R at the pole z_k.

Note 9 — Convergence of Sequences and Series

1. Crash Course in Complex Sequences and Series

Nothing here is new – it is all reducible to the real case via real and imaginary parts.

Some details.

$$z_n\to z\iff \big(\operatorname{Re}z_n\to\operatorname{Re}z\text{ and }\operatorname{Im}z_n\to\operatorname{Im}z\big)\iff |z_n-z|\to 0.$$

$$\sum_{n=1}^{\infty}z_n\text{ converges}\iff\Big\{\sum\operatorname{Re}z_n\text{ converges and }\sum\operatorname{Im}z_n\text{ converges}\Big\}.$$

Cauchy criterion for series

$\sum z_n$ converges if and only if

$$\forall\varepsilon>0\ \exists N\ \forall m>k>N:\left|\sum_{n=k}^{m}z_n\right|<\varepsilon.$$

Remark. $\sum_{n=1}^{\infty}z_n$ converges $\implies |z_n|\to 0$ (take $m=k$ in the Cauchy criterion).

Definition (absolute convergence)

$\sum z_n$ converges **absolutely** if $\sum|z_n|<\infty$.

Fact (absolute convergence $\iff$ bounded partial sums). $\sum z_n$ converges absolutely if and only if the partial sums $\sum_{n=0}^{N}|z_n|$ are bounded (as $N\to\infty$): since $|z_n|\geq 0$, these partial sums are non-decreasing, so (by the monotone convergence of real sequences) they converge – i.e. $\sum|z_n|<\infty$ – exactly when they are bounded above.

Important fact. $\sum|z_n|$ converges $\implies$ $\sum z_n$ converges (apply the Cauchy criterion to $\sum|z_n|$ and the triangle inequality $\big|\sum_{n=k}^{m}z_n\big|\leq\sum_{n=k}^{m}|z_n|$).

Another important fact. If $\sum z_n$ converges, then $\lim_{k\to\infty}\sum_{n=k}^{\infty}z_n=0$ (the tails of a convergent series vanish): apply the Cauchy criterion with $m\to\infty$.

2. Interlude: Supremum, Infimum, Limsup, and Liminf

We recall these notions for real sequences, since they will be needed for the radius of convergence of power series.

Definition (supremum and infimum)

Let $A\subseteq\mathbb{R}$. We say $a\in\mathbb{R}$ is an **upper bound** for A if $x\in A\implies x\leq a$; and $a:=\sup A$ if a is an upper bound and every other upper bound a' satisfies $a'\geq a$ (the *least* upper bound). If A has no upper bound, $\sup A:=+\infty$.

Symmetrically, a is a **lower bound** for A if $x\in A\implies x\geq a$; and $a:=\inf A$ if a is a lower bound and every other lower bound a' satisfies $a'\leq a$ (the *greatest* lower bound). If A has no lower bound, $\inf A:=-\infty$.

Example. $A=\{(-1)^n:n\in\mathbb{N}\}=\{-1,1\}$ – lower bounds are any $a\leq -1$, upper bounds are any $a\geq 1$.

Definition (limsup and liminf)

Let (a_n) be a real sequence. For each n let $R_n:=\{a_k:k\geq n\}$, $u_n:=\sup R_n$, $\ell_n:=\inf R_n$. Since $R_1\supseteq R_2\supseteq\cdots$, the sequence (u_n) is non-increasing and (ℓ_n) is non-decreasing; moreover $\ell_1\leq\ell_n\leq u_n\leq u_1$ for every n. So (u_n) is monotone and bounded below (by ℓ_1), and (ℓ_n) is monotone and bounded above (by u_1); hence both converge, and we define

$$\limsup_{n\to\infty}a_n:=\lim_n u_n=\inf_n u_n,\qquad \liminf_{n\to\infty}a_n:=\lim_n\ell_n=\sup_n\ell_n.$$

Equivalently,

$$\limsup_n a_n=\sup\big\{b:\#\{n:a_n>b\}\text{ is infinite}\big\}=\inf\big\{c:\#\{n:a_n>c\}\text{ is finite}\big\}.$$

Remark (unbounded sequences). If (a_n) is unbounded above, then every tail $R_n=\{a_k:k\geq n\}$ is also unbounded above (removing the finitely many terms with index $<n$ cannot bound it), so $u_n=\sup R_n=+\infty$ for *every* n, and hence $\limsup_n a_n=+\infty$. Likewise, if (a_n) is unbounded below, then $\ell_n=\inf R_n=-\infty$ for every n, so $\liminf_n a_n=-\infty$. These two phenomena are independent: e.g. $a_n=(-1)^n n$ is unbounded both above and below, with $\limsup a_n=+\infty$ and $\liminf a_n=-\infty$; while $a_n=n$ for n even, $a_n=0$ for n odd, is unbounded above but bounded below, with $\limsup a_n=+\infty$ but $\liminf a_n=0$ (finite) – note that being bounded below is *not* enough to force a finite $\liminf$: it is the odd terms, returning to 0 infinitely often, that keep $\ell_n=0$ for every n; a monotone sequence like $a_n=n$, by contrast, has $\liminf a_n=\limsup a_n=\lim a_n=+\infty$ as well (by the Fact below).

Example. $a_n=(-1)^n+\dfrac{\sin n}{n}$. Then $\limsup a_n=1$ and $\liminf a_n=-1$: for $b<1$, $\{n:a_n>b\}$ is infinite (it contains all sufficiently large even n, since $a_n=1+\sin(n)/n\to 1$ along evens), while for $b>1$ only finitely many n satisfy $a_n>b$.

Property (characterization of $\limsup$)

$\limsup_n a_n$ is the unique number ℓ such that:

(1) $\forall\varepsilon>0$: $\{n:a_n>\ell+\varepsilon\}$ is finite;

(2) $\forall\varepsilon>0$: $\{n:a_n>\ell-\varepsilon\}$ is infinite.

Proof. Let $\ell:=\limsup_n a_n=\inf_n u_n$, where $u_n=\sup R_n\downarrow\ell$.

(1) holds for ℓ. Since $u_n\downarrow\ell$, given $\varepsilon>0$ there is N with $u_N<\ell+\varepsilon$. Then

$a_k \le u_N < \ell + \varepsilon$ for every $k \ge N$, so $\{n : a_n > \ell + \varepsilon\} \subseteq \{1, \dots, N-1\}$ is finite.

(2) holds for ℓ. Given $\varepsilon > 0$, suppose toward a contradiction that $\{n : a_n > \ell - \varepsilon\}$ is finite; then there is N with $a_k \le \ell - \varepsilon$ for all $k \ge N$, so $u_N = \sup_{k \ge N} a_k \le \ell - \varepsilon < \ell$, contradicting $u_N \ge \inf_n u_n = \ell$. So $\{n : a_n > \ell - \varepsilon\}$ is infinite.

Uniqueness. Suppose ℓ' also satisfies (1) and (2). If $\ell' > \ell$, let $\varepsilon := (\ell' - \ell)/2 > 0$; then $\ell + \varepsilon = \ell' - \varepsilon$, and (1) for ℓ says $\{n : a_n > \ell + \varepsilon\}$ is finite, while (2) for ℓ' says $\{n : a_n > \ell' - \varepsilon\} = \{n : a_n > \ell + \varepsilon\}$ is infinite – a contradiction. If $\ell' < \ell$, let $\varepsilon := (\ell - \ell')/2 > 0$; then $\ell' + \varepsilon = \ell - \varepsilon$, and (1) for ℓ' says $\{n : a_n > \ell' + \varepsilon\}$ is finite, while (2) for ℓ says $\{n : a_n > \ell - \varepsilon\} = \{n : a_n > \ell' + \varepsilon\}$ is infinite – again a contradiction. So $\ell' = \ell$. □

Another characterization (of $\liminf$). By the symmetric argument (or applying the above to $-a_n$),

$$\liminf_n a_n = \inf\{b : \#\{n : a_n < b\} \text{ infinite}\} = \sup\{c : \#\{n : a_n < c\} \text{ finite}\},$$

and $\liminf_n a_n$ is likewise the unique number ℓ with $\{n : a_n < \ell - \varepsilon\}$ finite and $\{n : a_n < \ell + \varepsilon\}$ infinite for every $\varepsilon > 0$; the proof is the same, applied to $-a_n$.

Fact

$\lim_n a_n$ exists (possibly $\pm\infty$) if and only if $\limsup_n a_n = \liminf_n a_n$, in which case both equal $\lim_n a_n$.

Proof. Since $a_n \in R_n$, we have $\ell_n \le a_n \le u_n$ for every n.

($\Leftarrow$) If $\limsup_n a_n = \liminf_n a_n =: L$, then $u_n \to L$ and $\ell_n \to L$, so $\ell_n \le a_n \le u_n$ squeezes $a_n \to L$.

($\Rightarrow$) Suppose $a_n \to L$.

Case L finite. Fix $\varepsilon > 0$. Eventually $a_n < L + \varepsilon$, so for N large enough $u_N = \sup_{k \ge N} a_k \le L + \varepsilon$, giving $\limsup_n a_n = \inf_n u_n \le L + \varepsilon$. Likewise eventually $a_n > L - \varepsilon$, so $\ell_N \ge L - \varepsilon$ for N large, giving $\liminf_n a_n = \sup_n \ell_n \ge L - \varepsilon$. As $\varepsilon > 0$ was arbitrary, $\limsup_n a_n \le L \le \liminf_n a_n$; since always $\ell_n \le u_n$ forces $\liminf_n a_n \le \limsup_n a_n$, both inequalities are equalities and $\limsup_n a_n = \liminf_n a_n = L$.

Case $L = +\infty$. For every M, eventually $a_n > M$, so $\ell_N = \inf_{k \ge N} a_k \ge M$ for N large, giving $\liminf_n a_n = \sup_n \ell_n \ge M$; as M was arbitrary, $\liminf_n a_n = +\infty$, and then $\limsup_n a_n \ge \liminf_n a_n = +\infty$ forces $\limsup_n a_n = +\infty$ as well. The case $L = -\infty$ is symmetric (apply the above to $-a_n$). □

3. Pointwise versus Uniform Convergence

Let $f_n, f : K \to \mathbb{C}$, $K \subseteq \mathbb{C}$.

Definition (pointwise convergence)

$f_n \to f$ **pointwise** on K if

$$\forall z \in K\ \forall \varepsilon > 0\ \exists N(z, \varepsilon) : n > N(z, \varepsilon) \implies |f_n(z) - f(z)| < \varepsilon,$$

i.e., $|f_n(z) - f(z)| \to 0$ for each $z \in K$ separately.

Definition (uniform convergence)

$f_n \to f$ **uniformly** on K, written $f_n \rightrightarrows f$, if

$$\forall \varepsilon > 0\ \exists N(\varepsilon) : n > N(\varepsilon) \implies |f_n(z) - f(z)| < \varepsilon \quad \forall z \in K,$$

equivalently $\sup_{z \in K} |f_n(z) - f(z)| \to 0$.

Theorem

A uniform limit of a sequence of continuous functions is continuous.

Proof. Same proof as in the real case: fix $z_0 \in K$ and $\varepsilon > 0$; choose n with $\sup_{z \in K} |f_n(z) - f(z)| < \varepsilon/3$, then a neighbourhood of z_0 on which $|f_n(z) - f_n(z_0)| < \varepsilon/3$ (continuity of f_n); the triangle inequality gives $|f(z) - f(z_0)| < \varepsilon$ there. □

Example (pointwise but not uniform). On $[0, 1]$, $x^n \to \begin{cases} 0, & x < 1 \\ 1, & x = 1 \end{cases}$ pointwise, but the limit is discontinuous, so the convergence cannot be uniform.

4. The Weierstrass M-Test

Theorem (M-test for uniform convergence of series)

Let $K \subseteq \mathbb{C}$ and suppose $\sum_{n=1}^{\infty} M_n < \infty$ with $M_n \ge 0$. If $\forall z \in K\ \forall n$, $|f_n(z)| \le M_n$, then $\sum f_n(z)$ converges uniformly on K.

Proof. Since $\sum M_n$ converges, $\forall \varepsilon > 0\ \exists N$ such that $m > n > N \implies \sum_{k=n}^{m} M_k < \varepsilon$. Then for such m, n and any $z \in K$,

$$\left| \sum_{k=n}^{m} f_k(z) \right| \le \sum_{k=n}^{m} |f_k(z)| \le \sum_{k=n}^{m} M_k < \varepsilon,$$

so by the Cauchy criterion (uniformly in z), $\sum_{k=1}^{\infty} f_k(z) =: f(z)$ exists for every $z \in K$. Moreover

$$\left| f(z) - \sum_{k=1}^{n} f_k(z) \right| \le \sum_{k=n+1}^{\infty} M_k \to 0,$$

and the bound is independent of z, so the convergence is uniform. □

Note 10a — Power Series

1. Formal Power Series and the Convergence Lemma

Definition (power series)

Let $(a_n)_{n \ge 0}$ be a sequence of complex numbers. $\sum_{n=0}^{\infty} a_n z^n$ is a (formal) **power series**. Its **n-th partial sum** is the polynomial $S_n(z) := \sum_{k=0}^{n} a_k z^k$.

Lemma

Suppose that for some $r > 0$ the sequence $\left(|a_n| r^n\right)$ is bounded. Then for any z with $|z| < r$, $\sum a_n z^n$ converges. Moreover, if $r' < r$, the series converges uniformly on $\overline{B(0, r')}$.

Remark. The *uniform* convergence in the Lemma need not extend to all of $B(0, r)$, or even to $\overline{B(0, r)}$ itself: consider $\sum z^n$, where $r = 1$. The series converges (even absolutely) at every point of $B(0, 1)$, but not uniformly there; and it diverges at points of $\overline{B(0, 1)}$, e.g. at $z = 1$.

Proof. When $|z| \le r'$,

$$|a_n z^n| \le |a_n (r')^n| = |a_n| r^n \left(\frac{r'}{r}\right)^n < C \left(\frac{r'}{r}\right)^n,$$

using that $|a_n| r^n \le C$ for some constant C. Since $\sum C(r'/r)^k < \infty$ (a geometric series with ratio < 1), the Weierstrass M-test gives uniform convergence on $\overline{B(0, r')}$. □

Note. If $\left(|a_n| r^n\right)$ is *not* bounded, then for z with $|z| = r$, $\sum a_n z^n$ diverges, since some subsequence of $|a_n z^n| = |a_n| r^n$ is unbounded and so does not tend to 0.

Corollary (downward closure)

The set $S := \{t \ge 0 : (|a_n| t^n) \text{ is bounded}\}$ is **downward-closed**: if $s \in S$ and $0 \le t \le s$, then $t \in S$. Consequently, every $r' < \sup S$ satisfies $r' \in S$: since r' is not an upper bound for S, some $s \in S$ has $s > r'$, and downward closure gives $r' \in S$.

Proof. If $(|a_n| s^n)$ is bounded and $0 \le t \le s$, then $t^n \le s^n$ for every $n \ge 0$, so $|a_n| t^n \le |a_n| s^n$; hence $(|a_n| t^n)$ is bounded too. □

2. The Radius of Convergence

Theorem

For any power series $\sum a_n z^n$, let

$$R := \sup\left\{r : \text{the sequence } (|a_n| r^n) \text{ is bounded}\right\}.$$

Then:
(1) $\sum a_n z^n$ converges uniformly on $\overline{B(0, r)}$ for any $r < R$;
(2) $\sum a_n z^n$ diverges for any $|z| > R$.

Definition

R above is called the **radius of convergence** of the power series.

Proof. (1) If $r < R$, choose r' with $r < r' < R$; then $r' \in S$ by the Corollary above, so $(|a_n| (r')^n)$ is bounded, and the Lemma gives uniform convergence on $\overline{B(0, r)}$ since $r < r'$.

(2) If $|z| > R$, then $|z| \notin \{t : (|a_n| t^n) \text{ bounded}\}$, so $(|a_n| |z|^n)$ is unbounded, and the series diverges by the Note above. □

Cauchy–Hadamard Formula

$$R = \frac{1}{\limsup\limits_{n \to \infty} \sqrt[n]{|a_n|}} \qquad \left(\text{convention: } \tfrac{1}{0} = \infty,\ \tfrac{1}{\infty} = 0\right).$$

Proof. If $r < 1/\limsup_n \sqrt[n]{|a_n|}$, then $r \cdot \limsup_n \sqrt[n]{|a_n|} < 1$, so there exist $a < 1$ and N such that $n > N \implies r \sqrt[n]{|a_n|} < a$, i.e. $r^n |a_n| < a^n < 1$. So $(r^n |a_n|)$ is bounded, hence $r \le R$. This shows $R \ge 1/\limsup_n \sqrt[n]{|a_n|}$. Conversely, if $(r^n |a_n|)$ is bounded, say $r^n |a_n| \le M$ for all n, then $r \sqrt[n]{|a_n|} \le M^{1/n}$, so

$$\limsup_n r \sqrt[n]{|a_n|} \le \lim_{n \to \infty} M^{1/n} = 1, \qquad \text{so} \qquad r \le \frac{1}{\limsup_n \sqrt[n]{|a_n|}}.$$

So $1/\limsup_n \sqrt[n]{|a_n|}$ is an upper bound for $\{r : (|a_n| r^n) \text{ bounded}\}$, giving $R \le 1/\limsup_n \sqrt[n]{|a_n|}$. □

Summary. $\sum a_n z^n$ has radius of convergence $R = 1/\limsup_n \sqrt[n]{|a_n|}$; it converges uniformly on every disc $\{|z| \le r\}$ with $r < R$, and is continuous on $\{|z| < R\}$.

Note 10b — Analyticity of Power Series

1. Analyticity of Power Series

Lemma

Let $|z| \le r$, $|w| \le r$. Then $|z^n - w^n| \le n|z-w|r^{n-1}$.

Proof. $z^n - w^n = (z-w)(z^{n-1} + z^{n-2}w + \cdots + w^{n-1})$, so

$$|z^n - w^n| \le |z-w| \sum_{k=0}^{n-1} \underbrace{|w^k z^{n-1-k}|}_{\le r^{n-1}} \le n\, r^{n-1}|z-w|. \qquad \square$$

Theorem (analyticity of power series)

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence R. Then f is analytic on $B(0,R)$ and

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad \text{(termwise differentiation).}$$

Proof. Let $f_1(z) := \sum_{n=1}^{\infty} n a_n z^{n-1}$.

Step 1: f_1 has the same radius of convergence R. Let R_1 be the radius of convergence of $\sum_{n=1}^{\infty} n a_n z^{n-1}$. Since $|na_n| \ge |a_n|$, we get $R_1 \le R$. Conversely, if $r < R$, pick r_0 with $r < r_0 < R$; then

$$\lim_{n\to\infty} n|a_n| r^n = \lim_{n\to\infty} \left(|a_n| r_0^n\right) \cdot \left(n\left(\tfrac{r}{r_0}\right)^n\right) = 0,$$

since $|a_n| r_0^n$ is bounded (as $r_0 < R$) and $nq^n \to 0$ for $q = r/r_0 < 1$ (shown below). So $(n|a_n|r^n)$ is bounded, hence $r \le R_1$. As this holds for all $r < R$, $R \le R_1$, so $R_1 = R$.

Aside: $nq^n \to 0$ for $0 \le q < 1$. Write $q = 1/(1+\delta)$, $\delta > 0$. By the binomial theorem, $(1+\delta)^n > \binom{n}{2}\delta^2 = \frac{n(n-1)}{2}\delta^2$, so

$$nq^n < \frac{2n}{n(n-1)\delta^2} = \frac{2}{\delta^2(n-1)} \to 0.$$

Step 2: the difference quotient. Write $f(z) = S_n(z) + R_n(z)$, where $S_n(z) = \sum_{k=0}^{n} a_k z^k$ and $R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$; let $S_n'(z) = \sum_{k=1}^{n} k a_k z^{k-1}$ be the n-th partial sum of f_1. Fix $|z|, |z_0| < r < R$. Then

$$\frac{f(z)-f(z_0)}{z-z_0} - f_1(z_0) = \underbrace{\left(\frac{S_n(z)-S_n(z_0)}{z-z_0} - S_n'(z_0)\right)}_{\text{I}} + \underbrace{\left(S_n'(z_0) - f_1(z_0)\right)}_{\text{II}} + \underbrace{\frac{R_n(z)-R_n(z_0)}{z-z_0}}_{\text{III}}.$$

By the Lemma,

$$|\text{III}| = \left|\sum_{k=n+1}^{\infty} a_k \frac{z^k - z_0^k}{z-z_0}\right| \le \sum_{k=n+1}^{\infty} |a_k|\, k\, r^{k-1}.$$

Step 3: assembling the estimate. Fix $\varepsilon > 0$.

- Choose n large enough that $|S_n'(z_0) - f_1(z_0)| < \varepsilon/3$ (partial sums of f_1 converge, since f_1 has radius of convergence R), and $\sum_{k=n+1}^{\infty} k|a_k| r^{k-1} < \varepsilon/3$ (tail of a convergent series, as $r < R = R_1$).
- Then choose $\delta > 0$ such that $|z - z_0| < \delta \implies |\text{I}| < \varepsilon/3$, using differentiability of the polynomial S_n.

For $|z - z_0| < \delta$,

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f_1(z_0)\right| \le |\text{I}| + |\text{II}| + |\text{III}| < \varepsilon.$$

So $\lim_{z\to z_0} \dfrac{f(z)-f(z_0)}{z-z_0} = f_1(z_0)$, i.e. $f'(z_0) = f_1(z_0)$ for every $z_0 \in B(0,R)$. $\square$

Corollary (Taylor coefficients)

$f(z) = \sum a_n z^n$ is infinitely differentiable on $B(0,R)$, with

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\, a_n z^{n-k}, \qquad \text{and in particular} \qquad a_n = \frac{f^{(n)}(0)}{n!}.$$

Proof. Induction on k, applying the Theorem repeatedly to the (termwise-differentiated) power series, and setting $z = 0$ to read off $f^{(k)}(0) = k!\, a_k$. $\square$

Note 10-1 — Another Proof of Analyticity of Power Series

This note gives an alternative, more conceptual proof of the analyticity of power series (the Theorem of *Power Series*, §3), due to Weierstrass: instead of the direct ε-estimate splitting the difference quotient into three pieces, it expands $f(z_0 + h)$ via the binomial theorem and rearranges the resulting absolutely convergent double series. Section 1 develops the rearrangement facts needed; Section 2 gives the proof.

1. Preliminaries: Rearranging Series

Theorem (rearrangement of absolutely convergent series)

If $\sum_{n=0}^{\infty} z_n$ converges absolutely and $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection, then $\sum_{n=0}^{\infty} z_{\sigma(n)}$ converges absolutely, to the same sum: $\sum_n z_{\sigma(n)} = \sum_n z_n$.

Proof. First suppose $x_n \ge 0$. For any finite $F \subset \mathbb{N}$, taking $N := \max F$ gives $\sum_{n\in F} x_n \le \sum_{n=0}^{N} x_n \le \sum_n x_n$, so $\sup_F \sum_{n\in F} x_n \le \sum_n x_n$; and every partial sum $\sum_{n=0}^{N} x_n$ is itself of this form (with $F = \{0,\dots,N\}$), so in fact $\sum_n x_n = \sup_F \sum_{n\in F} x_n$ – a quantity manifestly independent of any ordering of the x_n. Hence $\sum_n x_{\sigma(n)} = \sup_F \sum_{n\in F} x_n = \sum_n x_n$ for every bijection σ.

For general complex z_n with $\sum |z_n| < \infty$: writing $\operatorname{Re} z_n = \operatorname{Re}^+ z_n - \operatorname{Re}^- z_n$ and $\operatorname{Im} z_n = \operatorname{Im}^+ z_n - \operatorname{Im}^- z_n$ (positive/negative parts), each of the four nonnegative sequences $\operatorname{Re}^{\pm} z_n$, $\operatorname{Im}^{\pm} z_n$ is $\le |z_n|$, hence sums absolutely and rearranges freely by the case above; combining the four via linearity of limits gives the same conclusion for $\sum z_n$ itself. $\square$

Theorem (summing a double series)

Let $(b_{n,k})_{n,k\ge 0}$ be complex numbers with $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} |b_{n,k}| < \infty$. Then, for any bijection $\tau : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$, the series $\sum_m b_{\tau(m)}$ converges absolutely to a value independent of τ, and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_{n,k} \;=\; \sum_{m} b_{\tau(m)} \;=\; \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} b_{n,k}.$$

Proof. Fix a bijection $\tau : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$. Applying the rearrangement theorem above to the nonnegative terms $|b_{\tau(m)}|$ shows $\sum_m |b_{\tau(m)}| = \sum_n \sum_k |b_{n,k}| < \infty$, independent of τ; so $\sum_m b_{\tau(m)}$ converges absolutely, and by the same theorem its value doesn't depend on τ either – call it S. Group the terms of $\sum_m b_{\tau(m)}$ "row by row": pass to the subsequence of partial sums obtained by including exactly the terms $b_{n,0},\dots,b_{n,k},\dots$ for $n = 0$, then all of row $n = 1$, and so on. A convergent series' partial sums converge, so any subsequence of partial sums – in particular one obtained by inserting parentheses – converges to the same limit S. This gives $S = \sum_n \left(\sum_k b_{n,k}\right)$. The same argument, grouping column by column instead, gives $S = \sum_k \left(\sum_n b_{n,k}\right)$. $\square$

2. Another Proof of Analyticity of Power Series

Theorem (analyticity of power series)

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence R. Then f is analytic on $B(0,R)$ and

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad \text{(termwise differentiation).}$$

Proof (Weierstrass, via rearrangement). Let $f_1(z) := \sum_{n=1}^{\infty} n a_n z^{n-1}$. Fix z_0 with $|z_0| < R$, and h with $0 < |h| < R - |z_0|$. By the binomial theorem, $(z_0 + h)^n = \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} h^k$, so

$$f(z_0+h) = \sum_{n=0}^{\infty} a_n (z_0+h)^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_{n,k}, \qquad b_{n,k} := \begin{cases} \binom{n}{k} a_n z_0^{n-k} h^k, & k \le n \\ 0, & k > n. \end{cases}$$

Since

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} |b_{n,k}| = \sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} \binom{n}{k} |z_0|^{n-k}|h|^k = \sum_{n=0}^{\infty} |a_n| \left(|z_0| + |h|\right)^n < \infty$$

(as $|z_0| + |h| < R$), the Double Series Theorem (Theorem above) lets us sum in the other order:

$$f(z_0+h) = \sum_{k=0}^{\infty} h^k\, c_k(z_0), \qquad c_k(z_0) := \sum_{n=k}^{\infty} \binom{n}{k} a_n z_0^{n-k},$$

and moreover $\sum_{k=0}^{\infty} |c_k(z_0)||h|^k \le \sum_{n=0}^{\infty} |a_n|(|z_0|+|h|)^n < \infty$ for every such h.

In particular $c_0(z_0) = f(z_0)$ and $c_1(z_0) = \sum_{n=1}^{\infty} n a_n z_0^{n-1} = f_1(z_0)$. Fix any ρ with $0 < \rho < R - |z_0|$; then $M := \sum_{k=2}^{\infty} |c_k(z_0)|\rho^{k-2} < \infty$ (a subseries, up to the factor ρ^{-2}, of the convergent $\sum_k |c_k(z_0)|\rho^k$), and for $0 < |h| \le \rho$,

$$\left|\frac{f(z_0+h)-f(z_0)}{h} - f_1(z_0)\right| = \left|\sum_{k=2}^{\infty} c_k(z_0) h^{k-1}\right| \le |h| \sum_{k=2}^{\infty} |c_k(z_0)||h|^{k-2} \le M|h| \xrightarrow[h\to 0]{} 0.$$

Hence $f'(z_0) = \lim_{h\to 0} \dfrac{f(z_0+h)-f(z_0)}{h} = f_1(z_0)$ for every $z_0 \in B(0,R)$. □

Remark. This argument shows more: re-expanded around any $z_0 \in B(0,R)$, f is itself given by the convergent power series $f(z_0+h) = \sum_k c_k(z_0)h^k$ in h (for $|h| < R-|z_0|$), so f automatically equals its own Taylor series at every point of $B(0,R)$ – the content of the *Corollary (Taylor coefficients)* in *Power Series*, §3 – with no need for a separate induction on the order of differentiation.

Note 11a — The Exponential, Euler's Formula, and the Logarithm

1. The Complex Exponential

Definition (complex exponential)

$$\exp(z) = e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots$$

For any $R > 0$, $\dfrac{R^n}{n!} \to 0$ as $n \to \infty$ (an easy exercise). Thus $\exp(z)$ exists for every $z \in \mathbb{C}$, and exp is entire. Differentiating term by term,

$$(e^z)' = \sum_{n=0}^{\infty} \left(\frac{z^n}{n!}\right)' = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = e^z.$$

The addition law. Fix $a \in \mathbb{C}$ and consider $f(z) := e^z e^{a-z}$. Then

$$f'(z) = e^z e^{a-z} + e^z \cdot (-e^{a-z}) = 0,$$

so $f \equiv f(0) = e^a$: for any $a, z \in \mathbb{C}$, $e^z e^{a-z} = e^a$. Setting $w := a - z$:

Addition law

$$e^z \cdot e^w = e^{z+w} \qquad \text{for all } z, w \in \mathbb{C}.$$

2. Euler's Formula

If $z = x + iy$, then $e^z = e^x e^{iy}$.

Observe. $\overline{e^z} = \sum \overline{z}^n/n! = e^{\bar z}$, so $|e^z|^2 = e^z\,\overline{e^z} = e^{z+\bar z} = e^{2x}$; hence $|e^z| = e^x$ and, in particular, $|e^{iy}| = 1$: e^{iy} lies on the unit circle.

Euler's Formula

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!} = \cos y + i \sin y =: \operatorname{cis}(y).$$

So $y \in \arg e^z$.

In particular, $e^{\pi i} = -1$.

Since $e^{2\pi i} = 1$, we get $e^z = e^{z+2k\pi i}$ for all $k \in \mathbb{Z}$: **the exponential is $2\pi i$-periodic.**

3. The Logarithm

e^w is the complex number with $|e^w| = e^{\operatorname{Re} w}$ and $\operatorname{Im} w \in \arg(e^w)$. To solve $z = e^w$ for w: if $z = 0$, there is no solution ($\log 0$ is undefined). If $z \neq 0$,

$\operatorname{Re} w = \log|z|$ (the usual real logarithm of $|z| > 0$), $\qquad \operatorname{Im} w \in \arg z$ (many possible values!).

Definition (log, multivalued)

$$\log z := \log|z| + i \arg z$$

is a **multivalued function** of $z \neq 0$.

Figure 4: The multivalued logarithm: the graph of $\operatorname{Im} \log z = \arg z$ over the z-plane. Each full turn around 0 climbs one level, by 2π, so the values of $\log z$ fill out an infinite spiral staircase of sheets.

Definition (principal value)

$$\operatorname{Log} z := \log|z| + i \operatorname{Arg} z, \qquad -\pi < \operatorname{Arg} z \le \pi,$$

is the **principal value** of the logarithm. Log is *not* continuous on $\{\operatorname{Re} z < 0,\ \operatorname{Im} z = 0\}$ (the negative real axis), where $\operatorname{Arg} z$ jumps between values near π and $-\pi$.

Remark. $e^{\operatorname{Log} z} = z$, but $\log(e^z) = \{z + 2\pi i k : k \in \mathbb{Z}\}$.

Definition (branch of the logarithm)

A continuous function $\ell(z)$ on a set K is called a **branch** of log if $\ell(z) \in \log z$ for every $z \in K$.

Example. $\ell(z) = \operatorname{Log} z$ is a branch on the slit plane $\{z : -\pi < \operatorname{Arg} z < \pi\} = \mathbb{C} \setminus \mathbb{R}_{\le 0}$. Other branches are obtained by choosing a different 2π-window for the argument – e.g. rotating the branch cut to lie along a different ray produces a different, equally valid, branch on the complementary slit plane.

Note 11b — Trigonometric, Hyperbolic, and Inverse Functions

1. The Logarithm (continued)

Theorem

If $\ell(z)$, a branch of log, is defined on $B(z_0, r)$, then $\ell(z)$ is analytic on $B(z_0, r)$ and $\ell'(z) = 1/z$.

Proof. Analyticity is a homework problem. Given analyticity, differentiate $e^{\ell(z)} = z$: by the chain rule and the inverse function rule,

$$\ell'(z) = \frac{1}{(e^w)'\big|_{w=\ell(z)}} = \frac{1}{e^{\ell(z)}} = \frac{1}{z}. \qquad \blacksquare$$

Theorem

On $\mathbb{D} = B(0,1)$,

$$\operatorname{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n.$$

Proof. Let $\ell(z) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$. Then

$$\ell'(z) = \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z} = \big(\operatorname{Log}(1+z)\big)'.$$

So $\ell(z) - \operatorname{Log}(1+z) \equiv \text{const} = \ell(0) - \operatorname{Log} 1 = 0$. □

2. Trigonometric Functions

Returning to Euler's equations, $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$ give

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Definition (sine and cosine)

$$\cos z := \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z := \frac{e^{iz} - e^{-iz}}{2i}.$$

Figure 5: The map $w = \sin z$. Since $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$, the half-strip $\{|\operatorname{Re} z| < \pi/2,\ \operatorname{Im} z > 0\}$ is mapped onto the upper half-plane: horizontal segments $\{\operatorname{Im} z = c\}$ go to half-ellipses with foci ± 1, and vertical rays $\{\operatorname{Re} z = c\}$ to branches of confocal hyperbolas.

Same power series as in the real case:

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \qquad \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!},$$

both 2π-periodic.

Basic identity.

$$\cos^2 z + \sin^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = 1.$$

Derivatives.

$$\cos' z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)' = i\,\frac{e^{iz} - e^{-iz}}{2} = -\sin z, \qquad \sin' z = \cos z.$$

Tangent.

$$\tan z := \frac{\sin z}{\cos z} = -i\,\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}, \qquad \pi\text{-periodic:}$$

$$\tan(z+\pi) = -i\,\frac{e^{iz+i\pi} - e^{-iz-i\pi}}{e^{iz+i\pi} + e^{-iz-i\pi}} = -i\,\frac{(-1)(e^{iz} - e^{-iz})}{(-1)(e^{iz} + e^{-iz})} = \tan z.$$

Its derivative is

$$\tan' z = \frac{4}{(e^{iz} + e^{-iz})^2} = \frac{1}{\cos^2 z}.$$

3. Hyperbolic Functions

Definition (hyperbolic sine and cosine)

$$\cosh z := \frac{e^z + e^{-z}}{2} = \cos(iz), \qquad \sinh z := \frac{e^z - e^{-z}}{2} = -i\sin(iz).$$

These satisfy $\cosh^2 z - \sinh^2 z = 1$, $\cosh' = \sinh$, $\sinh' = \cosh$, and

$$\coth z := \frac{\cosh z}{\sinh z}.$$

The pair $(\cosh t, \sinh t)$ parametrizes (one branch of) the hyperbola $x^2 - y^2 = 1$.

4. Inverse Functions

Arctangent (multivalued!). Solve $z = \tan w$:

$$z = -i\,\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} \implies e^{2iw} = \frac{1+iz}{1-iz} = \frac{i-z}{i+z} \implies \boxed{w = -\frac{i}{2}\log\left(\frac{i-z}{i+z}\right)}, \qquad z \neq \pm i.$$

As $z \to \infty$: $-\dfrac{i}{2}\log(-1) = -\dfrac{i}{2}(\pi i + 2\pi i k) = \dfrac{\pi}{2} + \pi k$, matching the real

arctangent's branch values at infinity.

Inverse hyperbolic sine. The same method gives

$$\operatorname{arcsinh} z = \log\left(z + \sqrt{z^2 + 1}\right).$$